General Certificate of Education (Advanced Level) – Grade 12

UNIT TEST – MARKING SCHEME

Combined Mathematics I & II

Notes for marking

- Marks in the right-hand column are awarded for each step shown. Part A questions total 25 marks; Part B questions total 150 marks.
- Method marks are given for a correct method even if a later arithmetic slip occurs; the answer mark is then lost. Follow-through is allowed only where stated.
- Any equivalent correct method earns full marks. Where a result is to be "shown", every step leading to the given answer must appear.
- For sketches, marks are for correct shape, correct key points (intercepts, vertices, asymptotes) and correct labelling.
- A correct final answer without supporting working receives at most the answer mark.

Combined Mathematics I

PART A

1. *[25 marks]*

$-7 \in \mathbb{Z}$ (integer, not natural); $0.375 = \dfrac{3}{8} \in \mathbb{Q}$; $\sqrt{49} = 7 \in \mathbb{N}$; $\sqrt{50} = 5\sqrt{2}$ and π are irrational. **(10)**

Let $x = 0.\dot{3}\dot{6}$. Then $100x = 36.\dot{3}\dot{6}$, so $99x = 36$ and $x = \dfrac{36}{99} = \dfrac{4}{11}$. **(5)**

Let $y = 1.2\dot{5} = 1.2555\ldots$ Then $100y - 10y = 125.\dot{5} - 12.\dot{5} = 113$, so $90y = 113$ and $y = \dfrac{113}{90}$. **(10)**

Total 25

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2. *[25 marks]*

$\sqrt{75} = 5\sqrt{3},\ \sqrt{12} = 2\sqrt{3},\ \sqrt{27} = 3\sqrt{3}$ **(5)**

$5\sqrt{3} - 4\sqrt{3} + 3\sqrt{3} = 4\sqrt{3}$ **(5)**

Multiply numerator and denominator by $\sqrt{5} + \sqrt{2}$: $\dfrac{(\sqrt{5} + \sqrt{2})^2}{5 - 2}$ **(5)**

$(\sqrt{5} + \sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10}$ **(5)**

$= \dfrac{7 + 2\sqrt{10}}{3}$ **(5)**

Total 25

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3. *[25 marks]*

$(2x-3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$ **(5)**

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$ **(5)**

$8a^3 - 27b^3 = (2a)^3 - (3b)^3 = (2a-3b)(4a^2+6ab+9b^2)$ **(5)**

$$\frac{2}{(x-1)(x+1)} - \frac{1}{x(x+1)} = \frac{2x-(x-1)}{x(x-1)(x+1)}$$ **(5)**

$$= \frac{x+1}{x(x-1)(x+1)} = \frac{1}{x(x-1)}, \quad x \neq 0, \pm 1$$ **(5)**

Total 25

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4. *[25 marks]*

$$225^\circ = 225 \times \frac{\pi}{180} = \frac{5\pi}{4}$$ **(5)**

$$\frac{7\pi}{6} = \frac{7 \times 180^\circ}{6} = 210^\circ$$ **(5)**

Arc length $s = r\theta = 6 \times \frac{2\pi}{3} = 4\pi$ cm **(5)**

Area $A = \frac{1}{2}r^2\theta$ **(5)**

$$= \frac{1}{2} \times 36 \times \frac{2\pi}{3} = 12\pi \text{ cm}^2$$ **(5)**

Total 25

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5. *[25 marks]*

$d = \sqrt{(4+2)^2 + (-5-3)^2}$ **(5)**

$= \sqrt{36+64} = 10$ **(5)**

Internal division $1:2$: $\left(\frac{1\cdot 7 + 2\cdot 1}{3}, \frac{1\cdot 4 + 2(-2)}{3}\right)$ **(5)**

$= (3, 0)$ **(5)**

External division $1:2$: $\left(\frac{1\cdot 7 - 2\cdot 1}{1-2}, \frac{1\cdot 4 - 2(-2)}{1-2}\right) = (-5, -8)$ **(5)**

Total 25

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6. *[25 marks]*

$$\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4} = -1$$

$$\sec\left(-\frac{\pi}{3}\right) = \frac{1}{\cos\frac{\pi}{3}} = 2$$ **(10)**

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$\sin^2\theta = 1 - \dfrac{9}{25} = \dfrac{16}{25}$; θ in the second quadrant so $\sin\theta > 0$: $\sin\theta = \dfrac{4}{5}$ **(10)**

$\tan\theta = \dfrac{4/5}{-3/5} = -\dfrac{4}{3}$ **(5)**

Total 25

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7. *[25 marks]*

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ **(5)**

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ (and its cyclic forms) **(5)**

(i) $c^2 = 49 + 25 - 2(7)(5)\cdot\dfrac{1}{2} = 39$, so $c = \sqrt{39}$ **(5)**

(ii) $b = \dfrac{a\sin B}{\sin A} = \dfrac{10\cdot\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}}$ **(5)**

$= \dfrac{10\sqrt{3}}{\sqrt{2}} = 5\sqrt{6}$ **(5)**

Total 25

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8. *[25 marks]*

$(f\circ g)(x) = f(x^2+1) = 3(x^2+1) - 2 = 3x^2 + 1$ **(5)**

$(g\circ f)(x) = (3x-2)^2 + 1 = 9x^2 - 12x + 5$ **(5)**

(i) Need $5 - x \geq 0$: domain $\{x : x \leq 5\} = (-\infty, 5]$ **(5)**

(ii) Need $x^2 - 9 > 0$, i.e. $(x-3)(x+3) > 0$ **(5)**

$x < -3$ or $x > 3$ **(5)**

Total 25

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9. *[25 marks]*

Sketch: line $y = x + 1$ for $x < 0$ (open circle at $(0, 1)$) **(5)**

horizontal segment $y = 1$ for $0 \leq x \leq 2$ **(5)**

line $y = 3 - x$ for $x > 2$ (open circle at $(2, 1)$, crossing the x-axis at $x = 3$) **(5)**

For $x < 0$, $f(x) < 1$; for $0 \leq x \leq 2$, $f(x) = 1$; for $x > 2$, $f(x) < 1$ **(5)**

Range $= (-\infty, 1]$ **(5)**

Total 25

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10. *[25 marks]*

$8^{2/3} = 4, \quad 16^{-1/4} = \dfrac{1}{2}, \quad 27^{-1/3} = \dfrac{1}{3}$ **(5)**

Value $= \dfrac{4 \times \frac{1}{2}}{\frac{1}{3}} = 6$ **(5)**

$9^x = 3^{2x}$ and $27^{x-1} = 3^{3x-3}$ **(5)**

$2x = 3x - 3$

$\Rightarrow \ x = 3$ **(5)**

$\log \dfrac{x^2 y^3}{\sqrt{z}} = 2\log x + 3\log y - \dfrac{1}{2}\log z$ **(5)**

Total 25

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11. *[25 marks]*

(1) $x + y + z = 6$, (2) $2x - y + z = 3$, (3) $x + 2y - z = 2$. (1)+(3): $2x + 3y = 8$ (4) **(5)**

(2)+(3): $3x + y = 5$ (5) **(5)**

From (5), $y = 5 - 3x$; substituting in (4): $2x + 15 - 9x = 8$

$\Rightarrow \ x = 1$ **(5)**

$y = 5 - 3 = 2$ **(5)**

From (1): $z = 6 - 1 - 2 = 3$. Check in (2): $2 - 2 + 3 = 3$ ✓ **(5)**

Total 25

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12. *[25 marks]*

Synthetic division by $x - 2$: coefficients $2, \ -3, \ 4, \ -5 \ \rightarrow \ 2, \ 1, \ 6, \ 7$ **(5)**

Quotient $2x^2 + x + 6$, remainder 7 (check: $p(2) = 16 - 12 + 8 - 5 = 7$) **(5)**

$q(3) = 27 - 18 - 15 + 6 = 0$, so by the factor theorem $(x - 3)$ is a factor **(5)**

$x^3 - 2x^2 - 5x + 6 = (x - 3)(x^2 + x - 2)$ **(5)**

$= (x - 3)(x + 2)(x - 1)$ **(5)**

Total 25

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13. *[25 marks]*

$2x^2 - 12x + 7 = 2(x^2 - 6x) + 7 = 2\left[(x - 3)^2 - 9\right] + 7$ **(5)**

$= 2(x - 3)^2 - 11$ **(5)**

Since $(x - 3)^2 \geq 0$, least value $= -11$, at $x = 3$ **(5)**

$\alpha + \beta = \dfrac{5}{2}, \quad \alpha\beta = \dfrac{1}{2}$ **(5)**

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$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \dfrac{25}{4} - 1 = \dfrac{21}{4}$

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \dfrac{25}{4} - 2 = \dfrac{17}{4}$ **(5)**

Total 25

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14. *[25 marks]*

(i) $\dfrac{x+5}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$\Rightarrow\ x+5 = A(x+2) + B(x-1)$ **(5)**

$x = 1$: $A = 2$

$x = -2$: $B = -1$. Result $\dfrac{2}{x-1} - \dfrac{1}{x+2}$ **(5)**

(ii) $3x^2 + 2x + 2 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$ **(5)**

$x = 2$: $18 = 9A$

$\Rightarrow\ A = 2$

$x = -1$: $3 = -3C$

$\Rightarrow\ C = -1$ **(5)**

Coefficient of x^2: $3 = A + B$

$\Rightarrow\ B = 1$. Result $\dfrac{2}{x-2} + \dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$ **(5)**

Total 25

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15. *[25 marks]*

(i) $(x-2)(x-3) > 0$ **(5)**

$x < 2$ or $x > 3$ **(5)**

(ii) $\dfrac{x-1}{2x+1} - 1 > 0$

$\Rightarrow\ \dfrac{-x-2}{2x+1} > 0$ **(5)**

$\Rightarrow \dfrac{x+2}{2x+1} < 0$; critical values $-2,\ -\dfrac{1}{2}$ **(5)**

$-2 < x < -\dfrac{1}{2}$ **(5)**

Total 25

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16. *[25 marks]*

(i) $2x + 1 \geq 5$ or $2x + 1 \leq -5$ **(5)**

$x \geq 2$ or $x \leq -3$ **(5)**

(ii) Case $x \geq -1$: $x + 1 > 3x + 7$

$\Rightarrow\ x < -3$, no solution in this case **(5)**

Case $x < -1$: $-x - 1 > 3x + 7$

$\Rightarrow\ x < -2$ **(5)**

Solution: $x < -2$ **(5)**

Total 25

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17. *[25 marks]*

Perimeter $2r + r\theta = 10 + 5\theta = 20$

$\Rightarrow\ \theta = 2$ rad **(5)**

Area $= \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(25)(2) = 25$ cm^2 **(5)**

Segment = sector − triangle $= \dfrac{1}{2}r^2\theta - \dfrac{1}{2}r^2\sin\theta$ **(5)**

$= \dfrac{1}{2}(100)\left(\dfrac{\pi}{3} - \sin\dfrac{\pi}{3}\right)$ **(5)**

$= 50\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}\right)$ cm^2 (≈ 9.06) **(5)**

Total 25

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18. *[25 marks]*

$AB^2 = 3^2 + 4^2 = 25,\ \ BC^2 = 4^2 + 3^2 = 25,\ \ AC^2 = 7^2 + 1^2 = 50$ **(5)**

$AB = BC$, so the triangle is isosceles **(5)**

$AB^2 + BC^2 = 50 = AC^2$, so $A\hat{B}C = 90^\circ$ (converse of Pythagoras) **(5)**

Locus: $(x-1)^2 + (y-2)^2 = (x-3)^2 + (y+4)^2$ **(5)**

$-2x - 4y + 5 = -6x + 8y + 25$

$\Rightarrow\ x - 3y = 5$ **(5)**

Total 25

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19. *[25 marks]*

$\sin(\pi+\theta) = -\sin\theta,\ \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta,\ \tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$ **(10)**

$\cos(2\pi-\theta) = \cos\theta,\ \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta,\ \cot(\pi-\theta) = -\cot\theta$ **(10)**

$$\text{Expression} = \frac{(-\sin\theta)(\sin\theta)(-\cot\theta)}{(\cos\theta)(\cos\theta)(-\cot\theta)} = -\tan^2\theta$$ **(5)**

Total 25

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20. *[25 marks]*

$$\sin 75^\circ = \sin 45^\circ\cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$ **(5)**

$$\tan 75^\circ = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$ **(5)**

$$= \frac{(\sqrt{3}+1)^2}{2} = 2+\sqrt{3}$$ **(5)**

$$\text{LHS} = 2 + 2(\tan\theta+\cot\theta) + \tan^2\theta + \cot^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta + \frac{2}{\sin\theta\cos\theta}$$ **(5)**

$\text{RHS} = \sec^2\theta + \operatorname{cosec}^2\theta + 2\sec\theta\operatorname{cosec}\theta$, the same. ■ **(5)**

Total 25

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21. *[25 marks]*

$$y = \frac{2x+1}{x-3}$$

$\Rightarrow\ x(y-2) = 3y+1$ **(5)**

$x = \frac{3y+1}{y-2}$ exists iff $y \neq 2$ (at $y=2$: $0=7$, impossible). Range $= \mathbb{R}\setminus\{2\}$ **(5)**

$f(a) = f(b)$

$\Rightarrow\ (2a+1)(b-3) = (2b+1)(a-3)$ **(5)**

$\Rightarrow -7a = -7b$

$\Rightarrow\ a = b$, so f is one-one **(5)**

$$f^{-1}(x) = \frac{3x+1}{x-2},\ \ x \neq 2$$ **(5)**

Total 25

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22. *[25 marks]*

$\alpha + \beta = 3, \quad \alpha\beta = 1$ **(5)**

(i) Sum $= \alpha + \beta + 4 = 7$

product $= \alpha\beta + 2(\alpha + \beta) + 4 = 11$ **(5)**

Equation: $x^2 - 7x + 11 = 0$ **(5)**

(ii) $\alpha^2 + \beta^2 = 9 - 2 = 7, \ \alpha^2\beta^2 = 1$ **(5)**

Equation: $x^2 - 7x + 1 = 0$ **(5)**

Total 25

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23. *[25 marks]*

If $k = 2$: expression $= 3 > 0$ for all x ✓ **(5)**

If $k \neq 2$: need $k - 2 > 0$ and discriminant $4(k-2)^2 - 12(k-2) < 0$, i.e.
$4(k-2)(k-5) < 0$ **(5)**

$2 < k < 5$. Hence $2 \leq k < 5$ **(5)**

Common root α: subtracting the equations, $2\alpha - k = 0$

$\Rightarrow \ \alpha = \dfrac{k}{2}$ **(5)**

$\dfrac{k^2}{4} - \dfrac{5k}{2} + k = 0$

$\Rightarrow \ k(k-6) = 0$

$\Rightarrow \ k = 6$; common root 3 **(5)**

Total 25

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24. *[25 marks]*

$p(1) = 0$: $a + b = 5$ **(5)**

$p(-1) = -4$: $-1 + a - b - 6 = -4$

$\Rightarrow \ a - b = 3$ **(5)**

$a = 4, \ b = 1$ **(5)**

$p(x) = x^3 + 4x^2 + x - 6 = (x-1)(x^2 + 5x + 6) = (x-1)(x+2)(x+3)$ **(5)**

Remainder $rx + s$: $r + s = p(1) = 0, \ -r + s = p(-1) = -4$

$\Rightarrow \ s = -2, \ r = 2$: remainder $2x - 2$ **(5)**

Total 25

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25. *[25 marks]*

(i) Let $y = 3^x > 0$: $3y^2 - 28y + 9 = 0$ **(5)**

$(3y - 1)(y - 9) = 0$

$\Rightarrow \; y = \dfrac{1}{3}$ or 9

$\Rightarrow \; x = -1$ or $x = 2$ **(5)**

(ii) Let $t = \log_2 x$ $(x > 0,\, x \neq 1)$; $\log_x 2 = \dfrac{1}{t}$: $\; t + \dfrac{1}{t} = \dfrac{5}{2}$ **(5)**

$2t^2 - 5t + 2 = 0$

$\Rightarrow \; (2t - 1)(t - 2) = 0$ **(5)**

$t = 2$

$\Rightarrow \; x = 4$

$t = \dfrac{1}{2}$

$\Rightarrow \; x = \sqrt{2}$ **(5)**

Total 25

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26. *[25 marks]*

(i) $\dfrac{x + 2}{5 - 2x} - 4 \geq 0$

$\Rightarrow \; \dfrac{9x - 18}{5 - 2x} \geq 0$

$\Rightarrow \; \dfrac{x - 2}{2x - 5} \leq 0$ **(5)**

$2 \leq x < \dfrac{5}{2} \;\; (x = \dfrac{5}{2}$ excluded) **(5)**

(ii) $\dfrac{x}{3} - \dfrac{1}{x + 2} \geq 0$

$\Rightarrow \; \dfrac{x^2 + 2x - 3}{3(x + 2)} \geq 0$

$\Rightarrow \; \dfrac{(x + 3)(x - 1)}{x + 2} \geq 0$ **(5)**

Sign table with critical values $-3,\; -2,\; 1$ **(5)**

$-3 \leq x < -2$ or $x \geq 1$ **(5)**

Total 25

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27. *[25 marks]*

$\cos\theta < 0$ in the third quadrant: $\cos\theta = -\sqrt{1-\frac{1}{9}} = -\frac{2\sqrt{2}}{3}$ **(5)**

$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) = \frac{4\sqrt{2}}{9}$ **(5)**

$\cos 2\theta = 1 - 2\sin^2\theta = \frac{7}{9}$

$\Rightarrow \tan 2\theta = \frac{4\sqrt{2}}{7}$ **(5)**

$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$ **(5)**

$= \frac{1+\frac{2\sqrt{2}}{3}}{-\frac{1}{3}} = -(3+2\sqrt{2})$ (negative, as $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$) **(5)**

Total 25

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28. *[25 marks]*

$f(x) = 2(x-2)^2 - 3$ **(5)**

f is symmetric about $x = 2$ and increasing for $x \geq 2$; least $k = 2$ **(5)**

Range $[-3, \infty)$ **(5)**

$y = 2(x-2)^2 - 3$

$\Rightarrow x - 2 = +\sqrt{\frac{y+3}{2}}$ (positive root since $x \geq 2$) **(5)**

$f^{-1}(x) = 2 + \sqrt{\frac{x+3}{2}},\ x \geq -3$ **(5)**

Total 25

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29. *[25 marks]*

Sketch of $y = |x| + 1$ (vertex $(0, 1)$) **(5)**

Sketch of $y = 2|x-1|$ (vertex $(1, 0)$, y-intercept 2) **(5)**

Intersections: $x \geq 1$: $x + 1 = 2x - 2$

$\Rightarrow x = 3$

$0 \leq x < 1$: $x + 1 = 2 - 2x$

$\Rightarrow x = \frac{1}{3}$

$x < 0$: no intersection **(5)**

$|x| + 1 < 2|x-1|$ where the first graph lies below the second **(5)**

$x < \frac{1}{3}$ or $x > 3$ **(5)**

Total 25

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30. *[25 marks]*

Sketch of $y = |2x - 3|$ (vertex $(\frac{3}{2}, 0)$, y-intercept 3) **(5)**

Sketch of $y = 3 - |x|$ (vertex $(0, 3)$, intercepts ± 3) **(5)**

Intersections: $0 \leq x \leq \frac{3}{2}$: $3 - 2x = 3 - x$

$\Rightarrow \; x = 0$

$x > \frac{3}{2}$: $2x - 3 = 3 - x$

$\Rightarrow \; x = 2$ **(5)**

$|2x - 3| < 3 - |x|$ for $0 < x < 2$ **(5)**

Replace x by $2x$: $|4x - 3| + 2|x| < 3$

$\iff \; 0 < 2x < 2$

$\iff \; 0 < x < 1$ **(5)**

Total 25

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PART B

Question 31 [150 marks]

(a) *[40]*

$(a+b\sqrt{3})^2 = a^2+3b^2+2ab\sqrt{3} = 7+4\sqrt{3}$ **(10)**

Equating rational and irrational parts: $a^2+3b^2=7,\ ab=2$ **(10)**

$a=2,\ b=1$ (taking the positive root) **(10)**

$\dfrac{1}{\sqrt{k+1}+\sqrt{k}} = \sqrt{k+1}-\sqrt{k}$, so the sum telescopes to $\sqrt{4}-\sqrt{1}=1$ **(10)**

(b) *[40]*

$a^3=(b+c)^3=b^3+c^3+3bc(b+c)=b^3+c^3+3abc$, so $a^3-b^3-c^3=3abc$ **(15)**

Take $a=3(x+2)$, $b=2x+3$, $c=x+3$; then $b+c=3x+6=a$, and $a^3=27(x+2)^3$ **(10)**

Expression $=3abc=9(x+2)(2x+3)(x+3)$ **(15)**

(c) *[30]*

$x^2+xy+y^2=(x+y)^2-xy=25-xy=19$

$\Rightarrow\ xy=6$ **(15)**

x, y are the roots of $t^2-5t+6=0$: $(x,y)=(2,3)$ or $(3,2)$ **(15)**

(d) *[40]*

Suppose $\sqrt{2}=\dfrac{p}{q}$, with p, q integers, $q\neq 0$, in lowest terms **(5)**

$p^2=2q^2$

$\Rightarrow\ p^2$ even $\Rightarrow p$ even; write $p=2m$ **(10)**

$q^2=2m^2$

$\Rightarrow\ q$ even, contradicting lowest terms; so $\sqrt{2}$ is irrational **(10)**

If $3+2\sqrt{2}=r\in\mathbb{Q}$, then $\sqrt{2}=\dfrac{r-3}{2}\in\mathbb{Q}$, a contradiction **(15)**

Question 32 [150 marks]

(a) *[35]*

$f(1)=f(-1)=1$, so f is not one-one **(10)**

-1 has no pre-image since $x^2\geq 0$, so f is not onto **(10)**

On $[0,\infty)$: $a^2=b^2$ with $a, b\geq 0$

$\Rightarrow\ a=b$ (one-one); for $y\geq 0$, $x=\sqrt{y}$ maps to y (onto) **(10)**

$f^{-1}(x)=\sqrt{x}$ **(5)**

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(b) *[35]*

$f(f(x)) = \dfrac{1+\frac{1+x}{1-x}}{1-\frac{1+x}{1-x}}$ **(10)**

$= \dfrac{2/(1-x)}{-2x/(1-x)} = -\dfrac{1}{x}$ **(10)**

$(f \circ f \circ f \circ f)(x) = (f \circ f)\left(-\dfrac{1}{x}\right) = -\dfrac{1}{-1/x} = x$ **(15)**

(c) *[40]*

$h(x) = x^2$ for $x \geq 0$ and $-x^2$ for $x < 0$; correct sketch (increasing curve through O) **(10)**

h is strictly increasing on $\mathbb{R}$, so $h(a) = h(b)$

$\Rightarrow\ a = b$ (one-one) **(10)**

For $y \geq 0$ take $x = \sqrt{y}$; for $y < 0$ take $x = -\sqrt{-y}$ (onto) **(10)**

$h^{-1}(x) = \sqrt{x}\ (x \geq 0),\ \ -\sqrt{-x}\ (x < 0)$ **(10)**

(d) *[40]*

$g(g(x)) = a^2x + ab + b = 4x - 9$

$\Rightarrow\ a^2 = 4,\ b(a+1) = -9$ **(10)**

$a = 2$

$\Rightarrow\ b = -3$

$a = -2$

$\Rightarrow\ b = 9$: $\ g(x) = 2x - 3$ or $g(x) = -2x + 9$ **(10)**

Replace x by $\dfrac{1}{x}$: $f\left(\dfrac{1}{x}\right) + 2f(x) = \dfrac{3}{x}$ **(10)**

Eliminating $f\left(\dfrac{1}{x}\right)$: $3f(x) = \dfrac{6}{x} - 3x$

$\Rightarrow\ f(x) = \dfrac{2}{x} - x$ **(10)**

Question 33 [150 marks]

(a) *[50]*

$\alpha + \beta = -p,\ \alpha\beta = q$ **(5)**

Sum of new roots $= (\alpha+\beta) + \dfrac{\alpha+\beta}{\alpha\beta} = -\dfrac{p(q+1)}{q}$ **(10)**

Product $= \alpha\beta + 2 + \dfrac{1}{\alpha\beta} = \dfrac{(q+1)^2}{q}$ **(10)**

Equation: $qx^2 + p(q+1)x + (q+1)^2 = 0$ **(5)**

Discriminant $= p^2(q+1)^2 - 4q(q+1)^2 = (q+1)^2(p^2 - 4q)$ **(10)**

≥ 0

$\iff\ \ q = -1$ or $p^2 \geq 4q$ **(10)**

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(b) *[70]*

$f(x) = 2(x-1)^2 + 3$: least value 3 at $x = 1$ **(10)**

$g(x) = -(x-k)^2 + k^2 - 3$: greatest value $k^2 - 3$ at $x = k$ **(10)**

(i) $k^2 - 3 < 3$

$\Rightarrow \ -\sqrt{6} < k < \sqrt{6}$ **(10)**

(ii) $f(x) - g(x) = 3x^2 - (4+2k)x + 8$ **(10)**

The graphs do not meet iff $f(x) - g(x) = 0$ has no real roots: $(4+2k)^2 - 96 < 0$

$\iff \ (k+2)^2 < 24$ **(10)**

$-2 - 2\sqrt{6} < k < -2 + 2\sqrt{6}$ **(10)**

(iii) If $\min f > \max g$ then $f(x) \geq \min f > \max g \geq g(x)$ for all x, so the graphs cannot meet. The converse fails because the minimum and maximum occur at different x; e.g. $k = 2.8$ lies in (ii) but not in (i). **(10)**

(c) *[30]*

$y(x^2+1) = x^2 + 2x + 3$

$\Rightarrow \ (y-1)x^2 - 2x + (y-3) = 0$ **(10)**

For real x (when $y \neq 1$): $4 - 4(y-1)(y-3) \geq 0$

$\Rightarrow \ y^2 - 4y + 2 \leq 0$; ($y = 1$ gives $x = -1$, which is real) **(10)**

$2 - \sqrt{2} \leq y \leq 2 + \sqrt{2}$ **(10)**

Question 34 [150 marks]

(a) *[30]*

$x^2 - (a+b)x + ab - c^2 = 0$ **(5)**

$\Delta = (a+b)^2 - 4ab + 4c^2 = (a-b)^2 + 4c^2 \geq 0$, so the roots are real **(15)**

The roots are equal iff $a = b$ and $c = 0$ **(10)**

(b) *[30]*

$\Delta = 4\lambda^2 + 4 > 0$, so the roots are real and distinct **(10)**

$\alpha + \beta = \dfrac{2\lambda}{k}, \ \alpha\beta = -\dfrac{1}{k^2}$ **(10)**

$\alpha^2 + \beta^2 = \dfrac{4\lambda^2}{k^2} + \dfrac{2}{k^2} = \dfrac{4\lambda^2 + 2}{k^2}$ **(10)**

(c) *[40]*

$y = \dfrac{x-1}{x+1}$

$\Rightarrow \ x = \dfrac{1+y}{1-y}$ **(10)**

Substitute in $ax^2 + bx + c = 0$ and multiply by $(1-y)^2$:
$a(1+y)^2 + b(1-y^2) + c(1-y)^2 = 0$ **(15)**

$(a-b+c)y^2 + 2(a-c)y + (a+b+c) = 0$ **(15)**

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(d) *[50]*

$(\alpha-\gamma)(\beta-\gamma)=\gamma^2+p\gamma+q$ **(10)**

Using $\gamma^2=-r\gamma-s$: $=(p-r)\gamma+(q-s)$; similarly $(\alpha-\delta)(\beta-\delta)=(p-r)\delta+(q-s)$ **(10)**

Product
$=(p-r)^2\gamma\delta+(p-r)(q-s)(\gamma+\delta)+(q-s)^2=(p-r)^2s-r(p-r)(q-s)+(q-s)^2$ **(10)**

$=(q-s)^2+(p-r)(ps-qr)$ **(10)**

Common root $\iff$ some factor is zero $\iff$ $(q-s)^2+(p-r)(ps-qr)=0$ **(10)**

Question 35 [150 marks]

(a) *[50]*

Division algorithm: $p(x)=(x-a)q(x)+R$, with R a constant **(10)**

Put $x=a$: $R=p(a)$ ■ **(10)**

$p(x)=(x-a)(x-b)Q(x)+\lambda x+\mu$ (remainder of degree ≤ 1) **(10)**

$p(a)=\lambda a+\mu,\ \ p(b)=\lambda b+\mu$ **(5)**

$\lambda=\dfrac{p(a)-p(b)}{a-b},\ \ \mu=\dfrac{a\,p(b)-b\,p(a)}{a-b}$ **(15)**

(b) *[50]*

$f(1)=2\lambda,\ \ f(2)=-\mu,\ \ f(3)=2\nu$ **(15)**

For x^5+kx^2: $f(1)=1+k$, $f(2)=32+4k$, $f(3)=243+9k$, so
$\lambda=\dfrac{1+k}{2},\ \ \mu=-(32+4k),\ \ \nu=\dfrac{243+9k}{2}$ **(15)**

Coefficient of x^2 in the remainder is $\lambda+\mu+\nu=0$ **(10)**

$\dfrac{1+k}{2}-32-4k+\dfrac{243+9k}{2}=90+k=0$
$\Rightarrow\ \ k=-90$ **(10)**

(c) *[50]*

$f(-1)=0$: $1-a+p+q=0$

$\Rightarrow\ \ a=1+p+q$ **(10)**

$g(1)=4$: $3+b-p-q=4$

$\Rightarrow\ \ b=1+p+q$ **(10)**

Hence $a=b$ **(5)**

$f(x)-f(-x)=2ax^3=0$ has a non-zero solution $\Rightarrow a=0$ **(10)**

Then $p+q=-1$ **(5)**

$g(-1)=1-b-q-2-p=-1-b-(p+q)=-1-0+1=0$, so $(x+1)$ is a factor of $g(x)$ **(10)**

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Question 36 [150 marks]

(a) *[30]*

RHS $= Ax^2 + (2A + B)x + (A + B + C)$ **(10)**

$A = 3,\ 2A + B = 5$

$\Rightarrow\ B = -1,\ A + B + C = -4$

$\Rightarrow\ C = -6$ **(20)**

(b) *[40]*

$p(2) = 0$: $16 + 4a + 2b - 6 = 0$

$\Rightarrow\ 2a + b = -5$ **(10)**

$p(-1) = -12$: $-2 + a - b - 6 = -12$

$\Rightarrow\ a - b = -4$ **(10)**

$a = -3,\ b = 1$ **(5)**

$p(x) = (x - 2)(2x^2 + x + 3)$ **(10)**

Discriminant of $2x^2 + x + 3$ is $1 - 24 < 0$, so $x = 2$ is the only real root **(5)**

(c) *[40]*

With $y = 7 - x$: $(y - 1)^4 + (y + 1)^4 = 16$ **(10)**

$2y^4 + 12y^2 + 2 = 16$ **(10)**

$y^4 + 6y^2 - 7 = 0$

$\Rightarrow\ (y^2 + 7)(y^2 - 1) = 0$ **(10)**

$y = \pm 1$ (real) $\Rightarrow x = 6$ or $x = 8$ **(10)**

(d) *[40]*

$h(1) = 1 - n + n - 1 = 0$, so $(x - 1)$ is a factor **(10)**

$h(x) = (x - 1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1 - n\right)$ **(15)**

Let $k(x) = x^{n-1} + \cdots + 1 - n$; $k(1) = n - n = 0$, so $(x - 1) \mid k(x)$ and hence $(x - 1)^2 \mid h(x)$ **(15)**

Question 37 [150 marks]

(a) *[40]*

Dividing by x^2 $(x \neq 0)$: $2\left(x^2 + \dfrac{1}{x^2}\right) + \left(x + \dfrac{1}{x}\right) - 11 = 0$ **(10)**

$x^2 + \dfrac{1}{x^2} = t^2 - 2$: $\ 2t^2 + t - 15 = 0$

$\Rightarrow\ (2t - 5)(t + 3) = 0$ **(10)**

$t = \dfrac{5}{2}$: $2x^2 - 5x + 2 = 0$

$\Rightarrow\ x = 2,\ \dfrac{1}{2}$ **(10)**

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$t = -3$: $x^2 + 3x + 1 = 0$

$\Rightarrow\ x = \dfrac{-3 \pm \sqrt{5}}{2}$ **(10)**

(b) *[40]*

$(x+2)(x+12) = x^2 + 14x + 24,\ \ (x+3)(x+8) = x^2 + 11x + 24$ **(10)**

$x = 0$ is not a root; dividing by x^2 with $t = x + \dfrac{24}{x}$: $(t+14)(t+11) = 4$ **(10)**

$t^2 + 25t + 150 = 0$

$\Rightarrow\ t = -10$ or $t = -15$ **(10)**

$x^2 + 10x + 24 = 0$

$\Rightarrow\ x = -4, -6$

$x^2 + 15x + 24 = 0$

$\Rightarrow\ x = \dfrac{-15 \pm \sqrt{129}}{2}$ **(10)**

(c) *[35]*

Form $\dfrac{A}{2t-1} + \dfrac{Bt + C}{4t^2 + 4t + 5}$ **(5)**

$t = \dfrac{1}{2}$: $8 = 8A$

$\Rightarrow\ A = 1$ **(10)**

$6t^2 + 7t + 3 - (4t^2 + 4t + 5) = 2t^2 + 3t - 2 = (2t-1)(t+2)$, so $Bt + C = t + 2$ **(15)**

Result: $\dfrac{1}{2t-1} + \dfrac{t+2}{4t^2 + 4t + 5}$ **(5)**

(d) *[35]*

$2x^3 + x^2 - x + 3 = 2(x^3 + x^2) + (-x^2 - x + 3)$ **(10)**

$-x^2 - x + 3 = Ax(x+1) + B(x+1) + Cx^2$ **(5)**

$x = 0$: $B = 3$

$x = -1$: $C = 3$

x^2-coefficient: $A + C = -1$

$\Rightarrow\ A = -4$ **(15)**

Result: $2 - \dfrac{4}{x} + \dfrac{3}{x^2} + \dfrac{3}{x+1}$ **(5)**

Question 38 [150 marks]

(a) *[35]*

With $u = x^2$: $\dfrac{1}{(u-1)(u+4)} = \dfrac{1}{5}\left[\dfrac{1}{u-1} - \dfrac{1}{u+4}\right]$ **(15)**

$\dfrac{1}{x^2 - 1} = \dfrac{1}{2}\left[\dfrac{1}{x-1} - \dfrac{1}{x+1}\right]$ **(10)**

Result: $\dfrac{1}{10(x-1)} - \dfrac{1}{10(x+1)} - \dfrac{1}{5(x^2+4)}$ **(10)**

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(b) *[35]*

$x^4 + 1 = (x^2 - 1)(x^2 + 1) + 2$ **(15)**

$= x^2 + 1 + \dfrac{2}{x^2 - 1} = x^2 + 1 + \dfrac{1}{x - 1} - \dfrac{1}{x + 1}$ **(20)**

(c) *[40]*

$1 = A(r + 1)(r + 2) + Br(r + 2) + Cr(r + 1)$: $\;r = 0$

$\Rightarrow\; A = \dfrac{1}{2}$

$r = -1$

$\Rightarrow\; B = -1$

$r = -2$

$\Rightarrow\; C = \dfrac{1}{2}$ **(20)**

$\dfrac{1}{2}\left[\dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}\right] = \dfrac{1}{2} \cdot \dfrac{(r+2) - r}{r(r+1)(r+2)} = \dfrac{1}{r(r+1)(r+2)}$ **(20)**

(d) *[40]*

Coefficient of $\dfrac{1}{x - 2}$ (cover-up) $= \dfrac{4 + 2p + q}{(1)(-1)} = 0$

$\Rightarrow\; 2p + q = -4$ **(10)**

Coefficient of $\dfrac{1}{x - 1} = \dfrac{1 + p + q}{(-1)(-2)} = 2$

$\Rightarrow\; p + q = 3$ **(10)**

$p = -7,\; q = 10$ **(10)**

$\dfrac{(x-2)(x-5)}{(x-1)(x-2)(x-3)} = \dfrac{x-5}{(x-1)(x-3)} = \dfrac{2}{x-1} - \dfrac{1}{x-3}$ **(10)**

Question 39 [150 marks]

(a) *[30]*

Let $\log_a b = y$: $b = a^y$

$\Rightarrow\; \log_c b = y \log_c a$

$\Rightarrow\; y = \dfrac{\log_c b}{\log_c a}$ **(10)**

Hence $\dfrac{1}{\log_k N} = \log_N k$ **(10)**

Sum $= \log_N 2 + \log_N 3 + \log_N 4 = \log_N 24 = \dfrac{1}{\log_{24} N}$ **(10)**

(b) *[35]*

$\log_9(xy^2) = \dfrac{\log_3 x + 2\log_3 y}{\log_3 9} = \dfrac{1}{2}\log_3 x + \log_3 y$ **(10)**

With $a = \log_3 x,\; b = \log_3 y$: $a + 2b = 1$, $ab = -3$ **(5)**

$(1 - 2b)b = -3$

$\Rightarrow\; 2b^2 - b - 3 = 0$

$\Rightarrow\; (2b - 3)(b + 1) = 0$ **(10)**

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$b = \frac{3}{2}, a = -2$: $x = \frac{1}{9},\ y = 3\sqrt{3}$

$b = -1, a = 3$: $x = 27,\ y = \frac{1}{3}$ **(10)**

(c) *[25]*

With $a = \log_2 x,\ b = \log_2 y$: $\frac{a}{3} + \frac{b}{2} = \frac{5}{2}$ and $\frac{b}{3} + \frac{a}{2} = \frac{5}{3}$, i.e. $2a + 3b = 15$ and $3a + 2b = 10$ **(10)**

$a = 0,\ b = 5$ **(10)**

$x = 1,\ y = 32$ **(5)**

(d) *[30]*

Divide by 16^x: $64\left(\frac{9}{16}\right)^x - 84\left(\frac{3}{4}\right)^x + 27 = 0$ **(10)**

With $t = \left(\frac{3}{4}\right)^x$: $64t^2 - 84t + 27 = 0$

$\Rightarrow\ (4t - 3)(16t - 9) = 0$ **(10)**

$t = \frac{3}{4}$

$\Rightarrow\ x = 1$

$t = \frac{9}{16}$

$\Rightarrow\ x = 2$ **(10)**

(e) *[30]*

$(a + b)^2 = a^2 + b^2 + 2ab = 9ab$ **(15)**

$\left(\frac{a+b}{3}\right)^2 = ab$

$\Rightarrow\ \log\frac{a+b}{3} = \frac{1}{2}\log(ab) = \frac{1}{2}(\log a + \log b)$ **(15)**

Question 40 [150 marks]

(a) *[35]*

$2r + r\theta = 40$

$\Rightarrow\ \theta = \frac{40 - 2r}{r}$ **(10)**

$A = \frac{1}{2}r^2\theta = \frac{1}{2}r(40 - 2r) = r(20 - r)$ **(10)**

$A = 100 - (r - 10)^2 \leq 100$; greatest area 100 cm^2 when $r = 10$ **(10)**

$\theta = \frac{40 - 20}{10} = 2$ rad **(5)**

(b) *[50]*

Area $= \frac{1}{2}(2r)^2\theta - \frac{1}{2}r^2\theta = \frac{3}{2}r^2\theta = 81$

$\Rightarrow\ \theta = \frac{54}{r^2}$ **(15)**

$p = r + r + r\theta + 2r\theta = 2r + 3r\theta = 2r + \frac{162}{r} = 2\left(r + \frac{81}{r}\right)$ **(10)**

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$r + \dfrac{81}{r} - 18 = \dfrac{(r-9)^2}{r} \geq 0$ for $r > 0$ **(10)**

$p \geq 36$, with equality when $r = 9$; least $p = 36$ m **(10)**

$\theta = \dfrac{54}{81} = \dfrac{2}{3}$ rad **(5)**

(c) *[35]*

Graph of $\sin x$ stretched by 2, shifted $\dfrac{\pi}{6}$ right and 1 up; correct shape on $[0, 2\pi]$ **(10)**

Greatest value 3 when $x - \dfrac{\pi}{6} = \dfrac{\pi}{2}$, i.e. $x = \dfrac{2\pi}{3}$ **(10)**

Least value -1 when $x - \dfrac{\pi}{6} = \dfrac{3\pi}{2}$, i.e. $x = \dfrac{5\pi}{3}$ **(10)**

Passes through $(0, 0)$ and $(2\pi, 0)$ **(5)**

(d) *[30]*

$x = 0$ is a solution **(5)**

For $|x| > \pi$, $\left|\dfrac{x}{\pi}\right| > 1 \geq |\sin x|$, so there are no solutions **(10)**

On $(0, \pi)$: the curve starts above the line (gradient $1 > \dfrac{1}{\pi}$) and ends below it ($\sin \pi = 0 < 1$); the curve is concave, so they cross exactly once **(10)**

Both functions are odd, so there is one solution in $(-\pi, 0)$; three solutions in total **(5)**

Question 41 [150 marks]

(a) *[40]*

$\tan 3\theta = \dfrac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta}$ **(10)**

$\Rightarrow \tan 3\theta - \tan 3\theta \tan 2\theta \tan \theta = \tan 2\theta + \tan \theta$, which rearranges to the result **(10)**

$\dfrac{1 - \cos\theta + \sin\theta}{1 + \cos\theta + \sin\theta} = \dfrac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$ **(10)**

$= \dfrac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} = \tan\dfrac{\theta}{2}$ **(10)**

(b) *[35]*

$\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \sin^2 60^\circ - \sin^2\theta = \dfrac{3}{4} - \sin^2\theta$ **(15)**

$4\sin\theta\left(\dfrac{3}{4} - \sin^2\theta\right) = 3\sin\theta - 4\sin^3\theta = \sin 3\theta$ **(10)**

With $\theta = 20^\circ$: $4\sin 20^\circ \sin 40^\circ \sin 80^\circ = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$, so the product $= \dfrac{\sqrt{3}}{8}$ **(10)**

(c) *[40]*

$\theta = 18^\circ$: $3\theta = 90^\circ - 2\theta$

$\Rightarrow \sin 3\theta = \cos 2\theta$ **(10)**

$3s - 4s^3 = 1 - 2s^2$

$\Rightarrow 4s^3 - 2s^2 - 3s + 1 = 0$ **(10)**

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$(s-1)(4s^2+2s-1)=0$ **(5)**

$s \neq 1$, so $s = \dfrac{-1 \pm \sqrt{5}}{4}$; $\sin 18^\circ > 0$

$\Rightarrow \ \sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$ **(10)**

$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - \dfrac{6-2\sqrt{5}}{8} = \dfrac{1+\sqrt{5}}{4}$ **(5)**

(d) *[35]*

$R\cos\alpha = \sqrt{3},\ R\sin\alpha = 1$

$\Rightarrow \ R = 2,\ \alpha = \dfrac{\pi}{6}$: $\ 2\cos\left(x + \dfrac{\pi}{6}\right)$ **(15)**

$1 \leq 3 + 2\cos\left(x + \dfrac{\pi}{6}\right) \leq 5$ **(10)**

Greatest value of the reciprocal is 1, least is $\dfrac{1}{5}$ **(10)**

Question 42 [150 marks]

(a) *[30]*

$\cos 4\theta = 2(2c^2-1)^2 - 1 = 8c^4 - 8c^2 + 1$ **(10)**

$\cos 3\theta = 4c^3 - 3c$ **(10)**

Difference $= 8c^4 - 4c^3 - 8c^2 + 3c + 1$ **(10)**

(b) *[30]*

$c^6 + s^6 = (c^2+s^2)^3 - 3c^2s^2(c^2+s^2) = 1 - 3s^2c^2$ **(10)**

$= 1 - \dfrac{3}{4}\sin^2 2\theta = 1 - \dfrac{3}{8}(1 - \cos 4\theta)$ **(10)**

$= \dfrac{5}{8} + \dfrac{3}{8}\cos 4\theta$: $\ a = \dfrac{5}{8},\ b = \dfrac{3}{8}$ **(10)**

(c) *[35]*

With $t = \tan x$: $\tan\left(\dfrac{\pi}{3} + x\right) = \dfrac{\sqrt{3}+t}{1-\sqrt{3}t},\ \tan\left(\dfrac{2\pi}{3} + x\right) = \dfrac{t-\sqrt{3}}{1+\sqrt{3}t}$ **(10)**

Their sum $= \dfrac{(\sqrt{3}+t)(1+\sqrt{3}t) + (t-\sqrt{3})(1-\sqrt{3}t)}{1-3t^2} = \dfrac{8t}{1-3t^2}$ **(10)**

Total $= t + \dfrac{8t}{1-3t^2} = \dfrac{9t-3t^3}{1-3t^2} = 3 \cdot \dfrac{3t-t^3}{1-3t^2} = 3\tan 3x$ **(15)**

(d) *[30]*

$2\sin\dfrac{\pi}{7}\cos\dfrac{2\pi}{7} = \sin\dfrac{3\pi}{7} - \sin\dfrac{\pi}{7},\ \ 2\sin\dfrac{\pi}{7}\cos\dfrac{4\pi}{7} = \sin\dfrac{5\pi}{7} - \sin\dfrac{3\pi}{7},$

$2\sin\dfrac{\pi}{7}\cos\dfrac{6\pi}{7} = \sin\pi - \sin\dfrac{5\pi}{7}$ **(15)**

Adding: $\ 2\sin\dfrac{\pi}{7} \cdot S = -\sin\dfrac{\pi}{7}$ **(10)**

$S = -\dfrac{1}{2}$ **(5)**

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(e) *[25]*

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2} = 2\cos\dfrac{C}{2}\cos\dfrac{A-B}{2}$ **(5)**

$\sin C = 2\sin\dfrac{C}{2}\cos\dfrac{C}{2} = 2\cos\dfrac{C}{2}\cos\dfrac{A+B}{2}$ **(10)**

$\text{Sum} = 2\cos\dfrac{C}{2}\left[\cos\dfrac{A-B}{2} + \cos\dfrac{A+B}{2}\right] = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$ **(10)**

Question 43 [150 marks]

(a) *[35]*

Numerator $= 2\sin 4x\cos 3x + 2\sin 4x\cos x$, denominator
$= 2\cos 4x\cos 3x + 2\cos 4x\cos x$ **(15)**

Ratio $= \tan 4x$ **(5)**

With $t = \tan\dfrac{\theta}{2}$: $\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = \dfrac{1+t}{1-t} = \dfrac{(1+t)^2}{1-t^2}$ **(5)**

$= \dfrac{1+t^2}{1-t^2} + \dfrac{2t}{1-t^2} = \sec\theta + \tan\theta$ **(10)**

(b) *[30]*

$\cos 45^\circ = 1 - 2\sin^2 22\tfrac{1}{2}^\circ$

$\Rightarrow\ \sin^2 22\tfrac{1}{2}^\circ = \dfrac{2-\sqrt{2}}{4}$; positive root $\sin 22\tfrac{1}{2}^\circ = \dfrac{1}{2}\sqrt{2-\sqrt{2}}$ **(10)**

$\tan 22\tfrac{1}{2}^\circ = \dfrac{1-\cos 45^\circ}{\sin 45^\circ} = \sqrt{2}-1$ **(10)**

$\tan 67\tfrac{1}{2}^\circ = \cot 22\tfrac{1}{2}^\circ = \dfrac{1}{\sqrt{2}-1} = \sqrt{2}+1$ **(10)**

(c) *[25]*

$\dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta + 2\cos^2\theta} = \dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)}$ **(20)**

$= \tan\theta$ **(5)**

(d) *[35]*

$\sin 2A + \sin 2B = 2\sin(A+B)\cos(A-B) = 2\sin C\cos(A-B)$ **(5)**

$\sin 2C = 2\sin C\cos C = -2\sin C\cos(A+B)$ **(10)**

$\text{Sum} = 2\sin C[\cos(A-B) - \cos(A+B)] = 4\sin A\sin B\sin C$ **(5)**

Sine rule: $\sin A = k$, $\sin B = \lambda k$, $\sin C = \mu k$ **(5)**

$\mu^2 \cdot 4k(\lambda k)(\mu k) = 4\lambda(\mu k)^3 = 4\lambda\sin^3 C$ **(10)**

(e) *[25]*

Correct sketches of $y = \sin x$ (period 2π) and $y = \cos 2x$ (period π) **(10)**

At intersections $1 - 2s^2 = s$

$\Rightarrow\ (2s-1)(s+1) = 0$ **(5)**

$x = \dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{3\pi}{2}$: three points **(10)**

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Question 44 [150 marks]

(a) *[60]*

$(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$

$\Rightarrow\ a^2+b^2+c^2 \geq ab+bc+ca$ **(10)**

$\frac{1}{2}(a+b+c)\cdot 2(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc$ **(10)**

(i) $a+b+c>0$ and the bracket is ≥ 0, so $a^3+b^3+c^3 \geq 3abc$ **(10)**

(ii) Apply (i) to $\sqrt[3]{a}$, $\sqrt[3]{b}$, $\sqrt[3]{c}$: $a+b+c \geq 3\sqrt[3]{abc}$ **(15)**

(iii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{\sqrt[3]{abc}}$; multiplying the two positive inequalities gives ≥ 9 **(15)**

(b) *[40]*

$\frac{2x}{(2x+1)(x+2)} - \frac{1}{x+1} \geq 0$ **(5)**

Numerator $2x(x+1)-(2x+1)(x+2) = -(3x+2)$ **(10)**

$\frac{3x+2}{(2x+1)(x+2)(x+1)} \leq 0$ **(5)**

Sign table, critical values $-2,\ -1,\ -\frac{2}{3},\ -\frac{1}{2}$ **(10)**

$-2<x<-1$ or $-\frac{2}{3} \leq x < -\frac{1}{2}$ **(10)**

(c) *[25]*

The denominator is always positive. Writing y for the expression:

$(1-y)x^2-(1+y)x+(1-y)=0$ **(5)**

Real x: $(1+y)^2-4(1-y)^2 \geq 0$ $(y=1$ gives $x=0)$ **(10)**

$(3y-1)(3-y) \geq 0$

$\Rightarrow\ \frac{1}{3} \leq y \leq 3$ **(10)**

(d) *[25]*

$1-\frac{6(3x+1)}{(x-1)(x-2)(x+3)} > 0$; numerator $x^3-7x+6-18x-6 = x(x-5)(x+5)$ **(10)**

$\frac{x(x-5)(x+5)}{(x-1)(x-2)(x+3)} > 0$ **(5)**

$x<-5,\ -3<x<0,\ 1<x<2$ or $x>5$ **(10)**

Question 45 [150 marks]

(a) *[30]*

$\frac{a}{b}+\frac{b}{a}-2 = \frac{(a-b)^2}{ab} \geq 0$ **(15)**

$\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b} = \frac{(a-b)^2}{ab(a+b)} \geq 0$ **(15)**

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(b) *[40]*

Expanding gives $4 + \sum \left(\frac{x}{y} + \frac{y}{x}\right)$ over the 6 pairs **(20)**

Each bracket ≥ 2 by (a), so the product $\geq 4 + 12 = 16$ **(20)**

(c) *[45]*

$2(ab+1) - (a+1)(b+1) = ab - a - b + 1 = (a-1)(b-1) > 0$ **(15)**

Similarly $2(cd+1) > (c+1)(d+1)$ **(5)**

$ab > 1,\ cd > 1$

$\Rightarrow\ 2(abcd+1) > (ab+1)(cd+1)$ **(10)**

$8(abcd+1) > 4(ab+1)(cd+1) > (a+1)(b+1)(c+1)(d+1)$ **(15)**

(d) *[35]*

$\frac{x^2 - 16x + 63}{x} \geq 0$ **(10)**

$\frac{(x-7)(x-9)}{x} \geq 0$ **(10)**

$0 < x \leq 7$ or $x \geq 9$ **(15)**

Question 46 [150 marks]

(a) *[40]*

Correct sketches (vertices $(\frac{3}{2}, -1)$ and $(0, 5)$) **(10)**

The inequality is equivalent to $|2x-3| - 1 < 5 - |3x|$ **(5)**

$x \geq \frac{3}{2}$: $5x - 3 < 6$

$\Rightarrow\ x < \frac{9}{5}$

$0 \leq x < \frac{3}{2}$: $x + 3 < 6$ (true)

$x < 0$: $3 - 5x < 6$

$\Rightarrow\ x > -\frac{3}{5}$ **(15)**

$-\frac{3}{5} < x < \frac{9}{5}$ **(5)**

Replace x by $2x$: $-\frac{3}{10} < x < \frac{9}{10}$ **(5)**

(b) *[40]*

Correct sketches **(10)**

$F(x) = x + 2|x-1| - 2|x+1| + 3$: $x \geq 1$: $x - 1$

$-1 \leq x < 1$: $3 - 3x$

$x < -1$: $x + 7$ **(15)**

$F > 0$: $x > 1,\ -1 \leq x < 1,\ -7 < x < -1$; i.e. $x > -7,\ x \neq 1$ **(10)**

Equation: $x = -7$ or $x = 1$ **(5)**

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(c) *[40]*

$x \geq \frac{1}{2}$: $3x - 3 \leq 5$

$\Rightarrow \; x \leq \frac{8}{3}$

$0 \leq x < \frac{1}{2}$: always true

$x < 0$: $3 - 3x \leq 5$

$\Rightarrow \; x \geq -\frac{2}{3}$ **(15)**

$-\frac{2}{3} \leq x \leq \frac{8}{3}$ **(5)**

$y = 3|x| - |6x - 3|$: $3 - 3x$ $(x \geq \frac{1}{2})$; $9x - 3$ $(0 \leq x < \frac{1}{2})$; $3x - 3$ $(x < 0)$; sketch **(10)**

Maximum $\frac{3}{2}$ at $x = \frac{1}{2}$, with the function decreasing on either side; exactly one solution iff $\alpha = \frac{3}{2}$ **(10)**

(d) *[30]*

Squaring: $(9 - b^2)x^2 - (6a - 4b)x + (a^2 - 4) < 0$ **(10)**

The solution set is a ray, so $9 - b^2 = 0$

$\Rightarrow \; b = \pm 3$ **(5)**

Then the inequality is $(6a - 4b)x > a^2 - 4$, which needs $6a - 4b > 0$ and $\frac{a^2 - 4}{6a - 4b} = \frac{4}{3}$ **(5)**

$b = 3$

$\Rightarrow \; \frac{a + 2}{6} = \frac{4}{3}$

$\Rightarrow \; a = 6$

$b = -3$

$\Rightarrow \; \frac{a - 2}{6} = \frac{4}{3}$

$\Rightarrow \; a = 10$ **(10)**

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Combined Mathematics II

PART A

1. *[25 marks]*

A scalar quantity has magnitude only (e.g. mass, time); a vector quantity has magnitude and direction (e.g. displacement, force) **(10)**

(i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$ **(5)**

(ii) $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ **(5)**

(iii) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \mathbf{0}$ **(5)**

Total 25

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2. *[25 marks]*

$|\mathbf{a}| = \sqrt{9 + 16} = 5$ **(5)**

$\hat{\mathbf{a}} = \dfrac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ **(5)**

$10\hat{\mathbf{a}} = 6\mathbf{i} + 8\mathbf{j}$ **(5)**

$-3\mathbf{i} + 9\mathbf{j} = -\dfrac{3}{2}(2\mathbf{i} - 6\mathbf{j})$, so they are parallel (opposite senses) **(10)**

Total 25

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3. *[25 marks]*

$2\mathbf{a} - 3\mathbf{b} = (4\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} + 9\mathbf{j}) = \mathbf{i} - 11\mathbf{j}$ **(10)**

$|2\mathbf{a} - 3\mathbf{b}| = \sqrt{1 + 121} = \sqrt{122}$ **(5)**

$\mathbf{a} + t\mathbf{b} = (2 + t)\mathbf{i} + (3t - 1)\mathbf{j}$; parallel to $\mathbf{i}$ iff $3t - 1 = 0$, so $t = \dfrac{1}{3}$ **(10)**

Total 25

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4. *[25 marks]*

$10\cos 30^\circ\,\mathbf{i} + 10\sin 30^\circ\,\mathbf{j} = 5\sqrt{3}\,\mathbf{i} + 5\mathbf{j}$ **(10)**

Horizontal component $50\cos 30^\circ = 25\sqrt{3}$ N **(10)**

Vertical component $50\sin 30^\circ = 25$ N **(5)**

Total 25

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5. *[25 marks]*

(i) $R = \sqrt{3^2 + 4^2} = 5$ N **(10)**

(ii) $R^2 = 5^2 + 8^2 + 2(5)(8)\cos 60^\circ = 129$ **(10)**

$R = \sqrt{129}$ N **(5)**

Total 25

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6. *[25 marks]*

$R^2 = P^2 + P^2 + 2P^2\cos 2\alpha = 2P^2(1 + \cos 2\alpha) = 4P^2\cos^2\alpha$

$\Rightarrow \; R = 2P\cos\alpha$ **(10)**

$\tan\phi = \dfrac{P\sin 2\alpha}{P + P\cos 2\alpha} = \tan\alpha$

$\Rightarrow \; \phi = \alpha$, so R bisects the angle **(5)**

$R = P$

$\Rightarrow \; 2\cos\alpha = 1$

$\Rightarrow \; \alpha = 60^\circ$, so the angle between the forces is 120° **(10)**

Total 25

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7. *[25 marks]*

$\overrightarrow{AB} = (5-2)\mathbf{i} + (3+1)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$ **(5)**

$|\overrightarrow{AB}| = 5$ **(5)**

The point dividing PQ in $2:3$: $\mathbf{p} + \dfrac{2}{5}(\mathbf{q} - \mathbf{p}) = \dfrac{1}{5}(3\mathbf{p} + 2\mathbf{q})$ **(15)**

Total 25

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8. *[25 marks]*

$|5\mathbf{i} - 12\mathbf{j}| = 13$ **(5)**

Unit vectors parallel: $\pm\dfrac{1}{13}(5\mathbf{i} - 12\mathbf{j})$ **(10)**

$-26 \cdot \dfrac{1}{13}(5\mathbf{i} - 12\mathbf{j}) = -10\mathbf{i} + 24\mathbf{j}$ **(10)**

Total 25

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9. *[25 marks]*

i: $2\lambda + \mu = 4$; **j**: $\lambda - 3\mu = 9$ **(10)**

Solving: $\lambda = 3,\ \mu = -2$ **(15)**

Total 25

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10. *[25 marks]*

$\mathbf{a} \cdot \mathbf{b} = 8 - 3 = 5$ **(5)**

$|\mathbf{a}| = \sqrt{13},\ |\mathbf{b}| = \sqrt{17}$ **(5)**

$\cos\theta = \dfrac{5}{\sqrt{221}}$ **(5)**

$(p\mathbf{i} + 2\mathbf{j}) \cdot (3\mathbf{i} - 4\mathbf{j}) = 3p - 8 = 0$

$\Rightarrow\ p = \dfrac{8}{3}$ **(10)**

Total 25

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11. *[25 marks]*

$\mathbf{i} \cdot \mathbf{i} = 1,\ \mathbf{i} \cdot \mathbf{j} = 0,\ \mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{i} \times \mathbf{i} = \mathbf{0}$ **(10)**

$(2\mathbf{i} + 3\mathbf{j}) \times (\mathbf{i} - \mathbf{j}) = 2\,\mathbf{i} \times \mathbf{i} - 2\,\mathbf{i} \times \mathbf{j} + 3\,\mathbf{j} \times \mathbf{i} - 3\,\mathbf{j} \times \mathbf{j}$ **(5)**

$= \mathbf{0} - 2\mathbf{k} - 3\mathbf{k} - \mathbf{0} = -5\mathbf{k}$ **(10)**

Total 25

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12. *[25 marks]*

$\mathbf{R} = (2 - 1 + 3)\mathbf{i} + (3 + 4 - 2)\mathbf{j} = 4\mathbf{i} + 5\mathbf{j}$ **(10)**

$|\mathbf{R}| = \sqrt{41}$ N **(5)**

$\mathbf{F} + (3\mathbf{i} - 2\mathbf{j}) + (-5\mathbf{i} + 4\mathbf{j}) = \mathbf{0}$

$\Rightarrow\ \mathbf{F} = 2\mathbf{i} - 2\mathbf{j}$ **(10)**

Total 25

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13. *[25 marks]*

With **i** east and **j** north: $X = 10 - 4 = 6$, $Y = 6 - 14 = -8$ **(10)**

$R = \sqrt{36 + 64} = 10$ N **(5)**

$\tan\theta = \dfrac{8}{6}$, so the direction is $\tan^{-1}\dfrac{4}{3}$ south of east **(10)**

Total 25

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14. *[25 marks]*

$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$, $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ **(10)**

D is the midpoint, so $\overrightarrow{DB} + \overrightarrow{DC} = \mathbf{0}$ **(5)**

$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$ **(5)**

$\overrightarrow{AB} + \overrightarrow{AC} + 2\overrightarrow{DA} = 2\overrightarrow{AD} - 2\overrightarrow{AD} = \mathbf{0}$ **(5)**

Total 25

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15. *[25 marks]*

Result: if $\mathbf{a}, \mathbf{b}$ are non-zero and non-parallel and $\lambda'\mathbf{a} + \mu'\mathbf{b} = \mathbf{0}$, then $\lambda' = \mu' = 0$ **(10)**

$2\lambda - 1 = 0$ and $\lambda + \mu - 3 = 0$ **(5)**

$\lambda = \dfrac{1}{2}$, $\mu = \dfrac{5}{2}$ **(10)**

Total 25

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16. *[25 marks]*

$\overrightarrow{AB} = -18\mathbf{a} + 12\mathbf{b}$, $\overrightarrow{BC} = 6\mathbf{a} - 4\mathbf{b}$ **(10)**

$\overrightarrow{AB} = -3\overrightarrow{BC}$: parallel with B common, so A, B, C are collinear **(5)**

$\overrightarrow{AC} = -12\mathbf{a} + 8\mathbf{b} = \dfrac{2}{3}\overrightarrow{AB}$, so $AC : CB = 2 : 1$ **(10)**

Total 25

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17. *[25 marks]*

$\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$ **(5)**

$\overrightarrow{DE} = -\mathbf{a}$ **(5)**

$\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$ **(5)**

$\overrightarrow{AD} = 2\mathbf{b}$ ($AD \parallel BC$ and $AD = 2BC$) **(5)**

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = 2\mathbf{b} - \mathbf{a}$ **(5)**

Total 25

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18. *[25 marks]*

Resolved part $= \dfrac{(3\mathbf{i} + 4\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})}{|\mathbf{i} + \mathbf{j}|} = \dfrac{7}{\sqrt{2}}$ **(10)**

$x\mathbf{i} + y\mathbf{j}$ with $3x + 4y = 0$ and $x^2 + y^2 = 1$ **(10)**

$\pm\dfrac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ **(5)**

Total 25

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19. *[25 marks]*

$|\mathbf{a} \pm 2\mathbf{b}|^2 = |\mathbf{a}|^2 \pm 4\,\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2$ **(10)**

Equal $\Rightarrow 8\,\mathbf{a}\cdot\mathbf{b} = 0$

$\Rightarrow\ \mathbf{a} \perp \mathbf{b}$ (angle 90°) **(5)**

$|\mathbf{a}+\mathbf{b}|^2 = 9 + 0 + 4 = 13$

$\Rightarrow\ |\mathbf{a}+\mathbf{b}| = \sqrt{13}$ **(10)**

Total 25

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20. *[25 marks]*

$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) = \mathbf{a}\times\mathbf{a} - \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} - \mathbf{b}\times\mathbf{b}$ **(10)**

$= \mathbf{0} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} - \mathbf{0} = 2\,\mathbf{b}\times\mathbf{a}$ **(5)**

Check: $(\mathbf{i}+\mathbf{j}) \times (\mathbf{i}-\mathbf{j}) = -\mathbf{i}\times\mathbf{j} + \mathbf{j}\times\mathbf{i} = -2\mathbf{k} = 2(\mathbf{j}\times\mathbf{i})$ ✓ **(10)**

Total 25

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21. *[25 marks]*

$P + Q = 17,\ P - Q = 7$

$\Rightarrow\ P = 12,\ Q = 5$ N **(10)**

At right angles: $\sqrt{144 + 25} = 13$ N **(5)**

$19P^2 = 4P^2 + 9P^2 + 12P^2\cos\theta$

$\Rightarrow\ \cos\theta = \dfrac{1}{2}$

$\Rightarrow\ \theta = 60^\circ$ **(10)**

Total 25

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22. *[25 marks]*

Take P along Ox: $X = P + 2P\cos 120^\circ = 0$ **(10)**

$Y = 2P\sin 120^\circ = \sqrt{3}P$ **(10)**

The resultant lies along Oy ($\perp P$), with magnitude $\sqrt{3}P$ **(5)**

Total 25

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23. *[25 marks]*

$X = P + 2P\cos 120^\circ + 3P\cos 240^\circ = -\dfrac{3}{2}P$ **(5)**

$Y = 2P\sin 120^\circ + 3P\sin 240^\circ = -\dfrac{\sqrt{3}}{2}P$ **(5)**

$R = \sqrt{\dfrac{9}{4} + \dfrac{3}{4}}\,P = \sqrt{3}P$ **(5)**

Third quadrant with $\tan\theta = \dfrac{1}{\sqrt{3}}$: angle 210° with Ox **(10)**

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Total 25

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24. *[25 marks]*

$\mathbf{a} \cdot \mathbf{b} = -5,\ |\mathbf{a}| = \sqrt{5},\ |\mathbf{b}| = \sqrt{10}$ **(5)**

$\cos\theta = -\dfrac{1}{\sqrt{2}}$

$\Rightarrow\ \theta = 135^\circ$ **(5)**

$\mathbf{a} + \lambda\mathbf{b} = (1 - 3\lambda)\mathbf{i} + (\lambda - 2)\mathbf{j} = \mu(-\mathbf{i} - 3\mathbf{j})$ **(5)**

$1 - 3\lambda = -\mu,\ \lambda - 2 = -3\mu$

$\Rightarrow\ \lambda - 2 = 3(1 - 3\lambda)$

$\Rightarrow\ \lambda = \dfrac{1}{2}$ **(10)**

Total 25

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25. *[25 marks]*

$\cos\theta = \dfrac{3}{2 \times 3} = \dfrac{1}{2}$

$\Rightarrow\ \theta = 60^\circ$ **(5)**

$|\mathbf{a} + \mathbf{b}|^2 = 4 + 2(3) + 9 = 19$

$\Rightarrow\ \sqrt{19}$ **(10)**

$\mathbf{a} \cdot (\mathbf{a} + \lambda\mathbf{b}) = 4 + 3\lambda = 0$

$\Rightarrow\ \lambda = -\dfrac{4}{3}$ **(10)**

Total 25

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26. *[25 marks]*

The midpoint D of BC has position vector $\dfrac{1}{2}(\mathbf{b} + \mathbf{c})$ **(5)**

$\overrightarrow{AG} = \dfrac{1}{3}(\mathbf{b} + \mathbf{c} - 2\mathbf{a}) = \dfrac{2}{3}\left(\dfrac{\mathbf{b} + \mathbf{c}}{2} - \mathbf{a}\right) = \dfrac{2}{3}\overrightarrow{AD}$ **(10)**

So G lies on AD with $AG : GD = 2 : 1$ **(5)**

By symmetry in $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the same holds for the other two medians **(5)**

Total 25

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27. *[25 marks]*

Resolving along Q: $2P\cos\theta + Q = 0$

$\Rightarrow\ 2PQ\cos\theta = -Q^2$ **(10)**

$R^2 = P^2 + Q^2 + 2PQ\cos\theta = P^2$

$\Rightarrow\ R = P$ **(5)**

New resultant$^2 = 4P^2 + Q^2 + 4PQ\cos\theta = 4P^2 - Q^2$ **(10)**

Total 25

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28. *[25 marks]*

Along P: $P + Q\cos\alpha = 0$ **(10)**

Perpendicular: $Q\sin\alpha = P$ **(5)**

$Q^2 = 2P^2$

$\Rightarrow\ Q = \sqrt{2}P$ **(5)**

$\cos\alpha < 0$, $\sin\alpha > 0$, $\tan\alpha = -1$

$\Rightarrow\ \alpha = 135^\circ$ **(5)**

Total 25

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29. *[25 marks]*

Opposite pairs combine: $(2-Q)$ along OA, $(7-3) = 4$ along OB, $(P-13)$ along OC **(10)**

Resolving perpendicular to OA: $4\sin 60^\circ + (P-13)\sin 120^\circ = 0$ **(5)**

$P = 9$ **(5)**

Along OA: $(2-Q) + 4\cos 60^\circ + (-4)\cos 120^\circ = 0$

$\Rightarrow\ Q = 6$ **(5)**

Total 25

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30. *[25 marks]*

$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\mathbf{c}$ **(5)**

$\overrightarrow{OP} = t\left(\mathbf{a} + \frac{1}{2}\mathbf{c}\right) = (1-s)\mathbf{a} + s\mathbf{c}$, with P on both OM and AC **(5)**

$\mathbf{a}, \mathbf{c}$ non-parallel: $t = 1 - s,\ \frac{t}{2} = s$ **(5)**

$t = \frac{2}{3},\ s = \frac{1}{3}$: $\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}$ **(5)**

$\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AC}$

$\Rightarrow\ AP : PC = 1 : 2$ **(5)**

Total 25

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PART B

Question 31 [150 marks]

(a) *[40]*

$\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{DE} = -\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{b}$, $\overrightarrow{FA} = \mathbf{a} - \mathbf{b}$ (opposite sides are equal and parallel; $\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}$) **(20)**

$\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$, $\overrightarrow{AD} = 2\mathbf{b}$, $\overrightarrow{AE} = 2\mathbf{b} - \mathbf{a}$, $\overrightarrow{AF} = \mathbf{b} - \mathbf{a}$ **(20)**

(b) *[35]*

D, E are the midpoints of AB, AC: $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$ **(10)**

$\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{BC}$ **(15)**

Hence $DE \parallel BC$ and $DE = \frac{1}{2}BC$ **(10)**

(c) *[40]*

In $\triangle ABC$: $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ (by (b)) **(15)**

In $\triangle ADC$: $\overrightarrow{SR} = \frac{1}{2}\overrightarrow{AC}$ **(15)**

$\overrightarrow{PQ} = \overrightarrow{SR}$: one pair of opposite sides is equal and parallel, so $PQRS$ is a parallelogram **(10)**

(d) *[35]*

Parallelogram $OABC$ with $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$; then $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ **(10)**

Midpoint of OB: $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ **(10)**

Midpoint of AC: $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ **(10)**

The midpoints coincide, so the diagonals bisect each other **(5)**

Question 32 [150 marks]

(a) *[40]*

If $\lambda \neq 0$, then $\mathbf{a} = -\frac{\mu}{\lambda}\mathbf{b}$, so $\mathbf{a} \parallel \mathbf{b}$: a contradiction **(20)**

Hence $\lambda = 0$; then $\mu\mathbf{b} = \mathbf{0}$ with $\mathbf{b} \neq \mathbf{0}$, so $\mu = 0$ **(20)**

(b) *[35]*

$\overrightarrow{AB} = 2\mathbf{a} - 3\mathbf{b}$, $\overrightarrow{AC} = (\lambda - 1)\mathbf{a} + 3\mathbf{b}$ **(10)**

Collinear $\Rightarrow \overrightarrow{AC} = k\overrightarrow{AB}$; by (a): $\lambda - 1 = 2k$ and $3 = -3k$ **(15)**

$k = -1$, $\lambda = -1$ **(10)**

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(c) *[35]*

$\overrightarrow{AP} = \dfrac{m}{m+n}\overrightarrow{AB}$ **(10)**

$\mathbf{p} = \mathbf{a} + \dfrac{m}{m+n}(\mathbf{b} - \mathbf{a})$ **(10)**

$= \dfrac{n\mathbf{a} + m\mathbf{b}}{m+n}$ **(15)**

(d) *[40]*

Resultant $= \lambda\mathbf{a} + \mu\mathbf{b}$ **(10)**

$= (\lambda + \mu) \cdot \dfrac{\lambda\mathbf{a} + \mu\mathbf{b}}{\lambda + \mu}$ **(10)**

By (c) with $m = \mu$, $n = \lambda$, $\dfrac{\lambda\mathbf{a} + \mu\mathbf{b}}{\lambda + \mu}$ is the position vector of C on AB with $AC : CB = \mu : \lambda$ **(20)**

Question 33 [150 marks]

(a) *[60]*

$\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{BC} = 4\mathbf{i} - 3\mathbf{j}$ **(10)**

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 12 - 12 = 0$, so $A\hat{B}C = 90^\circ$ **(10)**

$|\overrightarrow{AB}| = |\overrightarrow{BC}| = 5$, so the triangle is isosceles **(10)**

Area $= \dfrac{1}{2}(5)(5) = \dfrac{25}{2}$ **(10)**

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = 6\mathbf{i} - 2\mathbf{j}$ (check: $\overrightarrow{CD} = -3\mathbf{i} - 4\mathbf{j} = -\overrightarrow{AB}$) **(20)**

(b) *[50]*

$\mathbf{a} \cdot \mathbf{b} = 3 - 2 = 1$, $|\mathbf{a}| = \sqrt{5}$, $|\mathbf{b}| = \sqrt{10}$, $\cos\theta = \dfrac{1}{5\sqrt{2}}$ **(15)**

$\mathbf{a} \cdot \mathbf{b} = \lambda|\mathbf{b}|^2 + \mathbf{c} \cdot \mathbf{b} = 10\lambda$

$\Rightarrow \ \lambda = \dfrac{1}{10}$ **(20)**

$\mathbf{c} = \mathbf{a} - \dfrac{1}{10}\mathbf{b} = \dfrac{7}{10}\mathbf{i} + \dfrac{21}{10}\mathbf{j} = \dfrac{7}{10}(\mathbf{i} + 3\mathbf{j})$ (check $\mathbf{c} \cdot \mathbf{b} = 0$) **(15)**

(c) *[40]*

$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$ **(10)**

$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$ **(10)**

Adding gives $2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$ **(10)**

The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its four sides **(10)**

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Question 34 [150 marks]

(i) *[30]*

$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ **(15)**

$\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC} = -\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ **(15)**

(ii) *[30]*

N lies on CM: $\overrightarrow{ON} = \overrightarrow{OC} + \lambda\overrightarrow{CM} = 2\mathbf{a} + \lambda\left(-\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \left(2 - \frac{3}{2}\lambda\right)\mathbf{a} + \frac{1}{2}\lambda\mathbf{b}$ **(30)**

(iii) *[45]*

N lies on OB: $\overrightarrow{ON} = \mu\mathbf{b}$ **(10)**

$\mathbf{a}, \mathbf{b}$ non-parallel: $2 - \frac{3}{2}\lambda = 0$

$\Rightarrow \lambda = \frac{4}{3}$ **(15)**

$\overrightarrow{ON} = \frac{2}{3}\mathbf{b}$ **(10)**

$ON : NB = 2 : 1$ **(10)**

(iv) *[45]*

$\overrightarrow{OP} = \overrightarrow{OM} + k\mathbf{b} = \frac{1}{2}\mathbf{a} + \left(\frac{1}{2} + k\right)\mathbf{b}$ **(10)**

P on BC: $\overrightarrow{OP} = \mathbf{b} + \nu(2\mathbf{a} - \mathbf{b}) = 2\nu\mathbf{a} + (1-\nu)\mathbf{b}$ **(10)**

$2\nu = \frac{1}{2}$

$\Rightarrow \nu = \frac{1}{4}$

$\frac{1}{2} + k = \frac{3}{4}$

$\Rightarrow k = \frac{1}{4}$ **(15)**

$\overrightarrow{MP} = \frac{1}{4}\mathbf{b}$ **(10)**

Question 35 [150 marks]

(a) *[100]*

(i) $\overrightarrow{OD} = \mathbf{a} + \frac{1}{2}\mathbf{c}$ (since $\overrightarrow{AB} = \mathbf{c}$), so $\overrightarrow{OF} = \lambda\left(\mathbf{a} + \frac{1}{2}\mathbf{c}\right)$ **(15)**

$\overrightarrow{OE} = \mathbf{a} + \mathbf{c} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{a} + \mathbf{c}$, so $\overrightarrow{AE} = \mathbf{c} - \frac{1}{2}\mathbf{a}$ and $\overrightarrow{AF} = \mu\left(\mathbf{c} - \frac{1}{2}\mathbf{a}\right)$ **(15)**

(ii) $\overrightarrow{OF} = \mathbf{a} + \overrightarrow{AF} = \left(1 - \frac{\mu}{2}\right)\mathbf{a} + \mu\mathbf{c}$ **(10)**

Comparing coefficients: $\lambda = 1 - \frac{\mu}{2}, \ \frac{\lambda}{2} = \mu$ **(20)**

$\lambda = \frac{4}{5}, \ \mu = \frac{2}{5}$ **(20)**

$OF : FD = 4 : 1, \ AF : FE = 2 : 3$ **(20)**

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(b) *[50]*

With A as origin, $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$: $\overrightarrow{AD} = \frac{1}{3}(2\mathbf{b} + \mathbf{c})$ **(10)**

$\overrightarrow{AE} = \frac{1}{6}(2\mathbf{b} + \mathbf{c})$ **(10)**

$\overrightarrow{AF} = \mathbf{b} + t(\overrightarrow{AE} - \mathbf{b}) = \left(1 - \frac{2t}{3}\right)\mathbf{b} + \frac{t}{6}\mathbf{c}$ **(10)**

F on AC: $1 - \frac{2t}{3} = 0$

$\Rightarrow\ t = \frac{3}{2}$ **(10)**

$\overrightarrow{AF} = \frac{1}{4}\mathbf{c}$

$\Rightarrow\ AF : FC = 1 : 3$ **(10)**

Question 36 [150 marks]

(a) *[100]*

$\overrightarrow{OP} = \frac{1}{3}\mathbf{a},\ \overrightarrow{OQ} = \frac{3}{4}\mathbf{b}$ **(10)**

R on AQ: $\overrightarrow{OR} = \mathbf{a} + s\left(\frac{3}{4}\mathbf{b} - \mathbf{a}\right) = (1-s)\mathbf{a} + \frac{3s}{4}\mathbf{b}$ **(15)**

R on BP: $\overrightarrow{OR} = \mathbf{b} + t\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right) = \frac{t}{3}\mathbf{a} + (1-t)\mathbf{b}$ **(15)**

Comparing: $1 - s = \frac{t}{3},\ \frac{3s}{4} = 1 - t$ **(15)**

$s = \frac{8}{9},\ t = \frac{1}{3}$ **(15)**

$\overrightarrow{OR} = \frac{1}{9}\mathbf{a} + \frac{2}{3}\mathbf{b}$ **(10)**

$AR : RQ = 8 : 1$ and $BR : RP = 1 : 2$ **(20)**

(b) *[50]*

Parallelogram $OABC$, M the midpoint of AB; OM meets the diagonal AC at P **(10)**

$\overrightarrow{OP} = t\left(\mathbf{a} + \frac{1}{2}\mathbf{c}\right) = (1-s)\mathbf{a} + s\mathbf{c}$

$\Rightarrow\ t = \frac{2}{3},\ s = \frac{1}{3}$ **(25)**

$AP = \frac{1}{3}AC$, so P is a point of trisection of AC **(15)**

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Question 37 [150 marks]

(a) *[100]*

$\overrightarrow{OA} \cdot \overrightarrow{OB} = \alpha - 3 = 0$

$\Rightarrow \ \alpha = 3$ **(15)**

$\overrightarrow{AB} = 2\mathbf{i} - 4\mathbf{j}, \ \overrightarrow{AC} = \frac{1}{2}\mathbf{i} - \mathbf{j} = \frac{1}{4}\overrightarrow{AB}$, so A, B, C are collinear **(15)**

$AC : CB = 1 : 3$ **(10)**

$\overrightarrow{OD} = t\left(\frac{3}{2}\mathbf{i} + 2\mathbf{j}\right)$ **(10)**

$\overrightarrow{AD} = \left(\frac{3t}{2} - 1\right)\mathbf{i} + (2t - 3)\mathbf{j}$ **(10)**

$\overrightarrow{OD} \cdot \overrightarrow{AD} = \frac{3}{2}\left(\frac{3t}{2} - 1\right) + 2(2t - 3) = \frac{25t}{4} - \frac{15}{2} = 0$ **(15)**

$t = \frac{6}{5}$ **(10)**

$\overrightarrow{OD} = \frac{9}{5}\mathbf{i} + \frac{12}{5}\mathbf{j} = \frac{3}{5}(3\mathbf{i} + 4\mathbf{j})$ **(15)**

(b) *[50]*

$\overrightarrow{OA} = \mathbf{a}, \ \overrightarrow{OB} = -\mathbf{a}, \ \overrightarrow{OP} = \mathbf{p}$, with $|\mathbf{p}| = |\mathbf{a}|$ **(15)**

$\overrightarrow{PA} \cdot \overrightarrow{PB} = (\mathbf{a} - \mathbf{p}) \cdot (-\mathbf{a} - \mathbf{p}) = |\mathbf{p}|^2 - |\mathbf{a}|^2 = 0$ **(25)**

Hence $PA \perp PB$, i.e. $A\hat{P}B = 90^\circ$ **(10)**

Question 38 [150 marks]

(a) *[60]*

$\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$ **(15)**

Adding: $\mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} = 0$

$\Rightarrow \ \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0$ **(20)**

Hence $OC \perp AB$ **(10)**

Let O be the intersection of the altitudes from A and B in $\triangle ABC$; then $OC \perp AB$, so the third altitude also passes through O **(15)**

(b) *[50]*

Let $\overrightarrow{CB} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$; then $\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$ **(10)**

$c^2 = |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a} \cdot \mathbf{b}$ **(20)**

$= a^2 + b^2 - 2ab\cos C$ **(20)**

(c) *[40]*

$\mathbf{a} \times \mathbf{b} = a_1 b_2\,\mathbf{i} \times \mathbf{j} + a_2 b_1\,\mathbf{j} \times \mathbf{i}$ (since $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{0}$) **(15)**

$= (a_1 b_2 - a_2 b_1)\mathbf{k}$ **(10)**

$\mathbf{b} \times \mathbf{a} = (b_1 a_2 - b_2 a_1)\mathbf{k} = -\mathbf{a} \times \mathbf{b}$ **(15)**

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Question 39 [150 marks]

(a) *[35]*

By the sine rule in the triangle of components: $\dfrac{X}{\sin 45^\circ} = \dfrac{Y}{\sin 30^\circ} = \dfrac{20}{\sin 105^\circ}$ **(15)**

$X = \dfrac{20 \cdot \frac{\sqrt{2}}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 20(\sqrt{3}-1)$ N (along the 30° line) **(10)**

$Y = \dfrac{20 \cdot \frac{1}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 10(\sqrt{6}-\sqrt{2})$ N (along the 45° line) **(10)**

(b) *[40]*

$X = 1 + 2\cos 60^\circ + 3\cos 120^\circ + 4\cos 180^\circ = -\dfrac{7}{2}$ **(10)**

$Y = 2\sin 60^\circ + 3\sin 120^\circ = \dfrac{5\sqrt{3}}{2}$ **(10)**

$R = \sqrt{\dfrac{49}{4} + \dfrac{75}{4}} = \sqrt{31}$ N **(10)**

Second quadrant: angle $= 180^\circ - \tan^{-1}\dfrac{5\sqrt{3}}{7} \approx 128.9^\circ$ with Ox **(10)**

(c) *[35]*

Take the 2 N force along Ox, the 3 N at 120°, the 4 N at 240° **(5)**

$X = 2 - \dfrac{3}{2} - 2 = -\dfrac{3}{2},\ Y = \dfrac{3\sqrt{3}}{2} - 2\sqrt{3} = -\dfrac{\sqrt{3}}{2}$ **(15)**

$R = \sqrt{\dfrac{9}{4} + \dfrac{3}{4}} = \sqrt{3}$ N **(15)**

(d) *[40]*

$3\sqrt{2}(\cos 225^\circ\,\mathbf{i} + \sin 225^\circ\,\mathbf{j}) = -3\mathbf{i} - 3\mathbf{j}$ **(10)**

Sum of the given forces $= 5\mathbf{i} + 4\mathbf{j} - 3\mathbf{i} - 3\mathbf{j} = 2\mathbf{i} + \mathbf{j}$ **(15)**

Required force $= -2\mathbf{i} - \mathbf{j}$, magnitude $\sqrt{5}$ N, at $180^\circ + \tan^{-1}\dfrac{1}{2} \approx 206.6^\circ$ to Ox **(15)**

Question 40 [150 marks]

(a) *[60]*

$\mathbf{R} = (P-1)\mathbf{i} + (Q-3)\mathbf{j}$ **(15)**

Parallel to $\mathbf{i}+\mathbf{j}$ with magnitude $2\sqrt{2}$: $\mathbf{R} = \pm(2\mathbf{i} + 2\mathbf{j})$ **(15)**

$\mathbf{R} = 2\mathbf{i} + 2\mathbf{j}$: $P = 3,\ Q = 5$ **(15)**

$\mathbf{R} = -2\mathbf{i} - 2\mathbf{j}$: $P = -1,\ Q = 1$ **(15)**

(b) *[40]*

$13^2 = 7^2 + 8^2 + 2(7)(8)\cos\theta$

$\Rightarrow\ 169 = 113 + 112\cos\theta$ **(20)**

$\cos\theta = \dfrac{1}{2}$

$\Rightarrow\ \theta = 60^\circ$ **(20)**

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(c) *[50]*

In equilibrium, the resultant of the 4 N and 5 N forces is equal and opposite to the 6 N force **(20)**

$36 = 16 + 25 + 40\cos\theta$ **(15)**

$\cos\theta = -\dfrac{1}{8}$ **(15)**

Question 41 [150 marks]

(a) *[80]*

Parallelogram law: if two forces acting at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant is represented by the diagonal through that point **(15)**

With P along Ox: $X = P + Q\cos\alpha,\ \ Y = Q\sin\alpha$ **(15)**

$R^2 = X^2 + Y^2 = P^2 + 2PQ\cos\alpha + Q^2(\cos^2\alpha + \sin^2\alpha) = P^2 + Q^2 + 2PQ\cos\alpha$ **(20)**

$\tan\phi = \dfrac{Y}{X}$ **(15)**

$= \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$ **(15)**

(b) *[70]*

In the force triangle, the angle opposite Q is ϕ and the angle opposite P is $\alpha - \phi$:

$\dfrac{P}{\sin(\alpha-\phi)} = \dfrac{Q}{\sin\phi}$ **(15)**

$\dfrac{P-Q}{P+Q} = \dfrac{\sin(\alpha-\phi) - \sin\phi}{\sin(\alpha-\phi) + \sin\phi}$ **(10)**

$= \dfrac{2\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2} - \phi\right)}{2\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2} - \phi\right)} = \dfrac{\tan\left(\frac{\alpha}{2} - \phi\right)}{\tan\frac{\alpha}{2}}$ **(15)**

After the interchange, the resultant makes ϕ with the original line of Q, i.e. $\alpha - \phi$ with the original line of P **(10)**

The resultant turns through $\theta = (\alpha - \phi) - \phi = \alpha - 2\phi$ **(10)**

$\dfrac{\theta}{2} = \dfrac{\alpha}{2} - \phi$

$\Rightarrow\ \tan\dfrac{\theta}{2} = \dfrac{P-Q}{P+Q}\tan\dfrac{\alpha}{2}$ **(10)**

Question 42 [150 marks]

(a) *[50]*

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha,\ \ R'^2 = P^2 + Q^2 - 2PQ\cos\alpha$ **(20)**

Adding: $R^2 + R'^2 = 2(P^2 + Q^2)$ **(10)**

$P = Q$: $R^2 = 2P^2(1 + \cos\alpha),\ \ R'^2 = 2P^2(1 - \cos\alpha)$ **(10)**

$R^2 = 4R'^2$

$\Rightarrow\ 1 + \cos\alpha = 4 - 4\cos\alpha$

$\Rightarrow\ \cos\alpha = \dfrac{3}{5}$ **(10)**

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(b) *[60]*

(1) $R^2 = P^2 + Q^2 + 2PQc$

(2) $4R^2 = P^2 + 4Q^2 + 4PQc$

(3) $4R^2 = P^2 + Q^2 - 2PQc$ **(15)**

(2)−(3): $3Q^2 + 6PQc = 0$

$\Rightarrow\ 2PQc = -Q^2$ **(15)**

In (1): $R^2 = P^2$

$\Rightarrow\ R = P$ **(10)**

In (3): $4P^2 = P^2 + 2Q^2$

$\Rightarrow\ 3P^2 = 2Q^2$

$\Rightarrow\ P : Q = \sqrt{2} : \sqrt{3}$ **(15)**

$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$ **(5)**

(c) *[40]*

The resultant bisects the angle iff $\phi = \dfrac{\alpha}{2}$, i.e. $\tan\dfrac{\alpha}{2} = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$ **(10)**

$\iff (P + Q\cos\alpha)\sin\dfrac{\alpha}{2} = 2Q\sin\dfrac{\alpha}{2}\cos^2\dfrac{\alpha}{2} = Q(1+\cos\alpha)\sin\dfrac{\alpha}{2}$ **(20)**

$\iff P = Q\ \ (\text{as } \sin\dfrac{\alpha}{2} \neq 0)$ **(10)**

Question 43 [150 marks]

(a) *[50]*

$ABOF$ is a rhombus (O the centre), so $\overrightarrow{AB} + \overrightarrow{AF} = \overrightarrow{AO} = \dfrac{1}{2}\overrightarrow{AD}$ **(15)**

$\overrightarrow{AC} + \overrightarrow{AE} = 2\overrightarrow{AM}$, where M is the midpoint of CE; M lies on AD with $AM = \dfrac{3}{4}AD$, so $\overrightarrow{AC} + \overrightarrow{AE} = \dfrac{3}{2}\overrightarrow{AD}$ **(25)**

Total $= \dfrac{1}{2}\overrightarrow{AD} + \dfrac{3}{2}\overrightarrow{AD} + \overrightarrow{AD} = 3\overrightarrow{AD}$ **(10)**

(b) *[50]*

$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = (\mathbf{a} - \mathbf{o}) + (\mathbf{b} - \mathbf{o}) + (\mathbf{c} - \mathbf{o})$ **(15)**

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) - 3\mathbf{o}$ **(15)**

$= 3\,(\mathbf{g} - \mathbf{o}) = 3\overrightarrow{OG}$, since $\mathbf{g} = \dfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ **(20)**

(c) *[50]*

$\overrightarrow{AB} + 2\overrightarrow{BC} + 3\overrightarrow{CA} = (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) + \overrightarrow{BC} + 2\overrightarrow{CA} = \overrightarrow{BC} + 2\overrightarrow{CA}$ **(15)**

$= (\overrightarrow{BC} + \overrightarrow{CA}) + \overrightarrow{CA} = \overrightarrow{BA} + \overrightarrow{CA} = -(\overrightarrow{AB} + \overrightarrow{AC})$ **(15)**

$|\overrightarrow{AB} + \overrightarrow{AC}| = 2|\overrightarrow{AD}|$ (D the midpoint of BC) $= 2 \cdot \dfrac{\sqrt{3}}{2}a = \sqrt{3}a$ **(20)**

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Question 44 [150 marks]

(a) *[50]*

Forces: W vertically down, R normal to the plane, T at 30° to the line of greatest slope **(10)**

Along the plane: $T\cos 30^\circ = W\sin 30^\circ$

$\Rightarrow\ T = W\tan 30^\circ = \dfrac{W}{\sqrt{3}}$ **(20)**

Perpendicular: $R + T\sin 30^\circ = W\cos 30^\circ$

$\Rightarrow\ R = \dfrac{\sqrt{3}}{2}W - \dfrac{W}{2\sqrt{3}} = \dfrac{W}{\sqrt{3}}$ **(20)**

(b) *[50]*

Along the plane: $P\cos\alpha = W\sin\alpha$

$\Rightarrow\ P = W\tan\alpha$ **(25)**

Perpendicular: $R = W\cos\alpha + P\sin\alpha = W\cos\alpha + \dfrac{W\sin^2\alpha}{\cos\alpha} = W\sec\alpha$ **(25)**

(c) *[50]*

$3^2 + 4^2 = 5^2$, so the strings are perpendicular **(10)**

The $3a$ string makes angle θ_1 with the horizontal, $\cos\theta_1 = \dfrac{3}{5}$; the $4a$ string makes θ_2, $\cos\theta_2 = \dfrac{4}{5}$ **(10)**

Horizontal: $\dfrac{3}{5}T_1 = \dfrac{4}{5}T_2$

Vertical: $\dfrac{4}{5}T_1 + \dfrac{3}{5}T_2 = W$ **(15)**

$T_1 = \dfrac{4W}{5}$ ($3a$ string), $T_2 = \dfrac{3W}{5}$ ($4a$ string) **(15)**

Question 45 [150 marks]

(a) *[90]*

(i) Horizontal force F: vertical $T\cos\theta = W$, horizontal $T\sin\theta = F$ **(20)**

$T = W\sec\theta$, $F = W\tan\theta$ **(15)**

(ii) F perpendicular to the string: resolving along the string, $T = W\cos\theta$; perpendicular to it, $F = W\sin\theta$ **(25)**

$W\sin\theta = W\tan\theta\cos\theta < W\tan\theta$ for $0 < \theta < \dfrac{\pi}{2}$ **(15)**

So (ii) gives the smaller pulling force **(15)**

(b) *[60]*

The ring is smooth, so the tension T is the same in both parts **(10)**

By symmetry each part has length l and makes θ with the vertical, where $\sin\theta = \dfrac{d}{l}$ **(15)**

Vertical: $2T\cos\theta = W$ **(15)**

$T = \dfrac{W}{2\cos\theta} = \dfrac{Wl}{2\sqrt{l^2 - d^2}}$ **(20)**

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Question 46 [150 marks]

(a) *[100]*

Along the plane: $P\cos\theta = 20\sin 30^\circ = 10$

$\Rightarrow\ P = 10\sec\theta$ **(25)**

Perpendicular: $R + P\sin\theta = 20\cos 30^\circ = 10\sqrt{3}$ **(20)**

$R = 10\sqrt{3} - 10\tan\theta$ **(15)**

The particle stays in contact iff $R \geq 0$

$\iff\ \tan\theta \leq \sqrt{3}$ **(20)**

$0 \leq \theta \leq 60^\circ$ **(20)**

(b) *[50]*

Resultant $= \sqrt{8^2 + 15^2} = 17$ N **(20)**

It makes $\tan^{-1}\dfrac{15}{8}$ with the 8 N force **(15)**

The equilibrant is 17 N, equal and opposite to this resultant **(15)**

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