Simple Revision Notes

Combined Mathematics – Grade 12 Unit Test topics

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Pure Mathematics

1. Real numbers and surds

The idea

A **surd** is a root that does not simplify to a whole number, like $\sqrt{2}$. To **simplify**, pull out square factors. To **rationalise**, remove the root from the bottom of a fraction.

Steps: $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Bottom is $p - \sqrt{q}$? Multiply top and bottom by $p + \sqrt{q}$ (the "conjugate"), because $(p - \sqrt{q})(p + \sqrt{q}) = p^2 - q$, which has no root.

Worked example

$$\frac{6}{3-\sqrt{3}} = \frac{6(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})} = \frac{6(3+\sqrt{3})}{9-3} = 3+\sqrt{3}.$$

Recurring decimal $x = 0.\dot{3}\dot{6}$: two digits repeat, so $100x = 36.\dot{3}\dot{6}$. Subtract: $99x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$.

Common mistake

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. For example $\sqrt{9+16} = 5$, not $3+4$.

2. Functions

The idea

A function is a machine: put in x, get out exactly one $f(x)$.

- **Domain**: the x values allowed. **Range**: the outputs you actually get.
- **One-one**: different inputs always give different outputs.
- **Composite** $f \circ g$: do g first, then f.
- **Inverse** f^{-1}: undoes f. Only one-one functions have one.

Domain rules: no dividing by 0; nothing negative under $\sqrt{\ }$.
To find an inverse: write $y = f(x)$, make x the subject, then swap the letters.

Worked example

$f(x) = \dfrac{2x+1}{x-3}$. Write $y = \dfrac{2x+1}{x-3}$.

$y(x-3) = 2x+1 \;\Rightarrow\; yx - 2x = 3y+1 \;\Rightarrow\; x = \dfrac{3y+1}{y-2}$.

So $f^{-1}(x) = \dfrac{3x+1}{x-2}$, and the range of f is all reals except 2.

Common mistake

$f \circ g$ means g first. If $f(x) = 3x-2$ and $g(x) = x^2+1$, then $f(g(x)) = 3(x^2+1)-2$, not $(3x-2)^2+1$.

3. Quadratic functions and equations

The idea

$ax^2 + bx + c$. Three tools do almost every question:

- **Completing the square**: $a(x+p)^2 + q$ gives the vertex. The least value is q if $a > 0$.
- **Discriminant** $\Delta = b^2 - 4ac$: $\Delta > 0$ two real roots; $\Delta = 0$ equal roots; $\Delta < 0$ no real roots.
- **Roots** α, β: $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.

Positive for all x: need $a > 0$ **and** $\Delta < 0$.
New equation from roots: find the new sum S and product P, then write $x^2 - Sx + P = 0$.

Worked example

Roots of $x^2 - 3x + 1 = 0$ are α, β. Find the equation with roots α^2, β^2.
$\alpha + \beta = 3$, $\alpha\beta = 1$.
New sum $= \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 9 - 2 = 7$. New product $= (\alpha\beta)^2 = 1$.
Equation: $x^2 - 7x + 1 = 0$.

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Common mistake

If the x^2 coefficient contains k (e.g. $(k-2)x^2+\dots$), check $k=2$ separately: the expression stops being quadratic.

4. Polynomials

The idea

Remainder theorem: dividing $p(x)$ by $(x-a)$ leaves remainder $p(a)$.
Factor theorem: if $p(a)=0$, then $(x-a)$ is a factor.

To factorise a cubic: try $x=\pm1,\pm2,\pm3,\dots$ (factors of the constant term) until $p(a)=0$. Divide by $(x-a)$, then factorise the quadratic that is left.
Unknown constants: each piece of given information becomes one equation. Two unknowns need two equations.

Worked example

$p(x)=x^3-2x^2-5x+6$. $p(1)=1-2-5+6=0$, so $(x-1)$ is a factor.
Divide: $p(x)=(x-1)(x^2-x-6)=(x-1)(x-3)(x+2)$.

Common mistake

Dividing by $(2x-1)$: the remainder is $p\left(\frac{1}{2}\right)$, not $p(1)$. Set the divisor equal to 0 and solve for x.

5. Partial fractions

The idea

Split one fraction into simpler ones. Choose the form from the bottom (denominator):
$(x-a)\to\dfrac{A}{x-a}$; $\quad(x-a)^2\to\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$; $\quad(x^2+k)\to\dfrac{Ax+B}{x^2+k}$.

Steps: (1) If the top's degree $\geq$ the bottom's degree, divide first. (2) Write the form. (3) Multiply through by the denominator. (4) Substitute clever x values; compare coefficients for whatever is left.

Worked example

$\dfrac{x+5}{(x-1)(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{x+2}$, so $x+5=A(x+2)+B(x-1)$.
$x=1$: $6=3A\Rightarrow A=2$. $\quad x=-2$: $3=-3B\Rightarrow B=-1$.
Answer: $\dfrac{2}{x-1}-\dfrac{1}{x+2}$.

Common mistake

A repeated factor $(x+1)^2$ needs **two** terms, $\frac{B}{x+1}$ and $\frac{C}{(x+1)^2}$.

6. Indices and logarithms

The idea

$\log_a b = c$ means $a^c = b$. A log answers: "what power?"

Laws: $\log(xy) = \log x + \log y$; $\log \frac{x}{y} = \log x - \log y$; $\log x^n = n \log x$; change of base $\log_a b = \dfrac{\log_c b}{\log_c a}$.

Equations like $3^{2x} - \ldots$: put $y = 3^x$ to get a quadratic in y. Remember $y > 0$.
Mixed bases ($\log_2$ and $\log_4$): change everything to one base.

Worked example

$4^x - 3 \cdot 2^{x+1} + 8 = 0$. Let $y = 2^x$; then $4^x = y^2$ and $2^{x+1} = 2y$.
$y^2 - 6y + 8 = 0 \Rightarrow (y-2)(y-4) = 0 \Rightarrow y = 2$ or $4 \Rightarrow x = 1$ or 2.

Common mistake

Logs only exist for positive numbers. Reject answers that make anything inside a log ≤ 0.

7. Angular measure

The idea

π radians $= 180°$. With θ in radians: arc length $s = r\theta$, sector area $A = \frac{1}{2}r^2\theta$.

Sector perimeter $= 2r + r\theta$. Segment area = sector − triangle $= \frac{1}{2}r^2(\theta - \sin\theta)$.

Worked example

$r = 6$, $\theta = \frac{2\pi}{3}$: $s = 6 \cdot \frac{2\pi}{3} = 4\pi$; $A = \frac{1}{2} \cdot 36 \cdot \frac{2\pi}{3} = 12\pi$.

Common mistake

The formulas only work in radians. Convert degrees first: multiply by $\frac{\pi}{180}$.

8. Cartesian coordinates

The idea

Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Point dividing AB in the ratio $m : n$: internally $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$; externally, replace $+n$ by $-n$ everywhere.

Worked example

$A(1, -2)$, $B(7, 4)$, ratio $1 : 2$ internally: $\left(\frac{7+2}{3}, \frac{4-4}{3}\right) = (3, 0)$.

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9. Circular (trigonometric) functions

The idea

Signs: "All Students Take Coffee". Quadrant 1: all positive; Q2: sin; Q3: tan; Q4: cos.
Allied angles: with $\pi \pm \theta$ or $2\pi - \theta$ the function name stays the same; with $\frac{\pi}{2} \pm \theta$ or $\frac{3\pi}{2} \pm \theta$ it swaps (sin $\leftrightarrow$ cos, tan $\leftrightarrow$ cot). Then fix the sign using the quadrant.

Worked example

$\cos \frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right)$. This angle is in Q3, where cos < 0, so the value is $-\cos \frac{\pi}{3} = -\frac{1}{2}$.

Graphs: $y = a\sin(x - c) + d$ has amplitude a, is shifted right by c and up by d; maximum $d + a$, minimum $d - a$.

10. Trigonometric identities

The idea

Must-know formulas:

- $\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta$
- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B, \quad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\sin 2A = 2\sin A\cos A, \quad \cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
- $\sin 3A = 3\sin A - 4\sin^3 A, \quad \cos 3A = 4\cos^3 A - 3\cos A$
- $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}, \quad \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

Proving an identity: start from the messier side, change everything to sin and cos, then simplify towards the other side.
R-form: $a\cos x + b\sin x = R\cos(x - \alpha)$, with $R = \sqrt{a^2 + b^2}$. It always lies between $-R$ and R.

Worked example

$\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$, since $R = \sqrt{3+1} = 2$ and $\tan\alpha = \frac{1}{\sqrt{3}}$. So it ranges from -2 to 2.

Common mistake

When you find $\cos\theta$ from $\sin\theta$, choose the sign of the square root using the quadrant.

11. Sine rule and cosine rule

The idea

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. Use it when you know a side and its opposite angle.
Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$. Use it with two sides and the angle between them, or with all three sides.

Worked example

$a = 3, b = 5, c = 7$: $\cos C = \dfrac{9 + 25 - 49}{30} = -\frac{1}{2}$, so $C = 120°$.

12. Inequalities – step by step

The idea

An inequality asks: **which x values make this true?** The answer is usually a range, like $x < 6$, not a single number.

A. Linear. Solve exactly like an equation, with one exception: if you multiply or divide by a **negative** number, flip the sign.

Worked example

$-2x < 6$. Divide by -2 and flip: $x > -3$.
Check with $x = 0$: $-2(0) = 0 < 6$ ✓. Check with $x = -5$: $10 < 6$? No, so -5 is correctly excluded.

B. Quadratic. Example: $x^2 - 5x + 6 > 0$.

1. Factorise: $(x - 2)(x - 3) > 0$.
2. The "critical values" are 2 and 3. Mark them on a number line; this splits it into three pieces.
3. Test one number from each piece:
   $x = 0$: $(-2)(-3) = +6 > 0$ ✓ $x = 2.5$: $(0.5)(-0.5) < 0$ × $x = 4$: $(2)(1) > 0$ ✓
4. Answer: $x < 2$ or $x > 3$.

Shortcut (when the x^2 term is positive): "> 0" means **outside** the roots; "< 0" means **between** them.

C. Fractions (rational). Never multiply both sides by something with x in it: if it is negative the sign should flip, and you cannot tell. Use a sign table instead. Example: $\dfrac{x-1}{2x+1} > 1$.

1. Move everything to one side: $\dfrac{x-1}{2x+1} - 1 > 0$.
2. Make one fraction: $\dfrac{x-1-(2x+1)}{2x+1} = \dfrac{-x-2}{2x+1} > 0$.
3. Critical values: top $= 0$ at $x = -2$; bottom $= 0$ at $x = -\frac{1}{2}$.
4. Test each piece: $x = -3$: $\frac{1}{-5} < 0$ ×; $x = -1$: $\frac{-1}{-1} = 1 > 0$ ✓; $x = 0$: $\frac{-2}{1} < 0$ ×.
5. Answer: $-2 < x < -\frac{1}{2}$.

The bottom can never be 0, so $x = -\frac{1}{2}$ is never included, even in a "$\geq$" question.

D. Modulus. $|x|$ is the distance from 0, so it is never negative. For a positive number a:

- $|x| < a$ means x is close to 0: $-a < x < a$.
- $|x| > a$ means x is far from 0: $x < -a$ or $x > a$.

Worked example

$|x - 2| < 3$: $-3 < x - 2 < 3$, so $-1 < x < 5$.
$|2x + 1| \geq 5$: $2x + 1 \geq 5$ or $2x + 1 \leq -5$, so $x \geq 2$ or $x \leq -3$.

Harder modulus questions (cases). Remove the modulus by splitting at the point where the inside equals 0. Example: $|x + 1| > 3x + 7$; the inside is 0 at $x = -1$.

- **Case** $x \geq -1$: $|x + 1| = x + 1$, so $x + 1 > 3x + 7 \Rightarrow x < -3$. That disagrees with $x \geq -1$, so there is no solution in this case.
- **Case** $x < -1$: $|x + 1| = -(x + 1)$, so $-x - 1 > 3x + 7 \Rightarrow x < -2$. This agrees with $x < -1$ ✓.
- Answer: $x < -2$.

Graph method: sketch both sides (V-shapes), find where they cross, and read off where the left side is lower (for $<$).

E. Proving inequalities. Key fact: a square is never negative, $(a - b)^2 \geq 0$. Move everything to one side and show it equals a square divided by something positive.

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Worked example

Show $\dfrac{a}{b}+\dfrac{b}{a}\geq 2$ for $a,b>0$.

$\dfrac{a}{b}+\dfrac{b}{a}-2=\dfrac{a^2+b^2-2ab}{ab}=\dfrac{(a-b)^2}{ab}\geq 0$ ✓

Common mistake

Four habits that cost marks: (1) multiplying by an x-expression; (2) including a value that makes the denominator 0; (3) not checking each case against its own range; (4) writing "and" when you mean "or".

Applied Mathematics

13. Vectors: the basics

The idea

A **vector** has size and direction (force, displacement). A **scalar** has size only (mass, time).
In $\mathbf{i},\mathbf{j}$ form: $\mathbf{a}=3\mathbf{i}+4\mathbf{j}$ means 3 across and 4 up. Its size is $|\mathbf{a}|=\sqrt{3^2+4^2}=5$; the unit vector is $\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}$.

Adding: add the $\mathbf{i}$ parts and the $\mathbf{j}$ parts separately. **Chain rule:** $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.
Parallel: $\mathbf{a}\parallel\mathbf{b}$ when $\mathbf{a}=k\mathbf{b}$ for some number k.

Worked example

$\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$. $\quad -3\mathbf{i}+9\mathbf{j}=-\frac{3}{2}(2\mathbf{i}-6\mathbf{j})$, so the two are parallel.

14. Position vectors and the key result

The idea

The position vector of A is $\mathbf{a}=\overrightarrow{OA}$, and $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.
The point dividing AB in the ratio $m:n$ is $\dfrac{n\mathbf{a}+m\mathbf{b}}{m+n}$; the midpoint is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.
Key result: if $\mathbf{a},\mathbf{b}$ are not parallel and $\lambda\mathbf{a}+\mu\mathbf{b}=\mathbf{0}$, then $\lambda=0$ and $\mu=0$.

Where two lines cross: write the point in two ways, one for each line, each with its own unknown. Make the $\mathbf{a}$ parts equal and the $\mathbf{b}$ parts equal, then solve.

Worked example

In parallelogram $OABC$, OM (with M the midpoint of AB) meets AC at P.
On OM: $\overrightarrow{OP}=t(\mathbf{a}+\frac{1}{2}\mathbf{c})$. On AC: $\overrightarrow{OP}=(1-s)\mathbf{a}+s\mathbf{c}$.
Matching: $t=1-s$ and $\frac{t}{2}=s$, so $t=\frac{2}{3}$, $s=\frac{1}{3}$. Hence $AP:PC=1:2$.

Collinear: show $\overrightarrow{AB}=k\overrightarrow{BC}$; the shared point B then puts all three on one line.

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15. Scalar (dot) product and vector product

The idea

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. In components: $(a_1\mathbf{i}+a_2\mathbf{j})\cdot(b_1\mathbf{i}+b_2\mathbf{j}) = a_1b_1+a_2b_2$.
Perpendicular $\iff \mathbf{a}\cdot\mathbf{b}=0$. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.
Vector product (properties only): $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{a}\times\mathbf{a}=\mathbf{0}$, $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

Worked example

Angle between $\mathbf{i}-2\mathbf{j}$ and $-3\mathbf{i}+\mathbf{j}$: dot product $=-3-2=-5$; sizes $\sqrt{5}$ and $\sqrt{10}$.
$\cos\theta = \dfrac{-5}{\sqrt{50}} = -\dfrac{1}{\sqrt{2}}$, so $\theta = 135°$.

Common mistake

Use $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\,\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$. Do not add the sizes directly.

16. Forces at a point

The idea

Two forces P, Q with angle α between them:

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha, \qquad \tan\phi = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

(ϕ is the angle between R and P.)

Many forces – the resolving method (use this most):

1. Choose two perpendicular directions (x and y).
2. Add up all the x-parts: $X = \sum F\cos\theta$. Add up all the y-parts: $Y = \sum F\sin\theta$.
3. Resultant $R = \sqrt{X^2+Y^2}$, at angle $\tan^{-1}\frac{Y}{X}$ (check the quadrant).
4. **Equilibrium** means $X=0$ and $Y=0$.

Worked example

10 N east, 6 N north, 4 N west, 14 N south.
$X = 10-4 = 6$; $Y = 6-14 = -8$; $R = \sqrt{36+64} = 10$ N, at $\tan^{-1}\frac{4}{3}$ south of east.

Particle on a smooth plane: resolve **along** the slope and **perpendicular** to it. The weight W splits into $W\sin\alpha$ down the slope and $W\cos\alpha$ into the plane.

Common mistake

Draw the diagram first, mark every force (weight, tension, reaction), then resolve. Most lost marks come from a missing force.

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